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TECHNICAL REPORT GL-88-20

THERMODYNAMICS OF FRICTIONAL MATERIALS

Report 1

CONSTITUTIVE THEORY OF SOILS WITH DILATANT CAPABILITY

by

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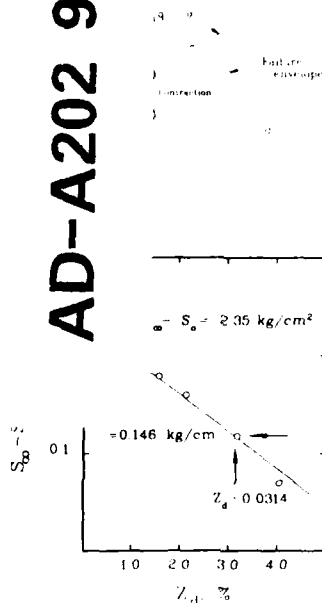
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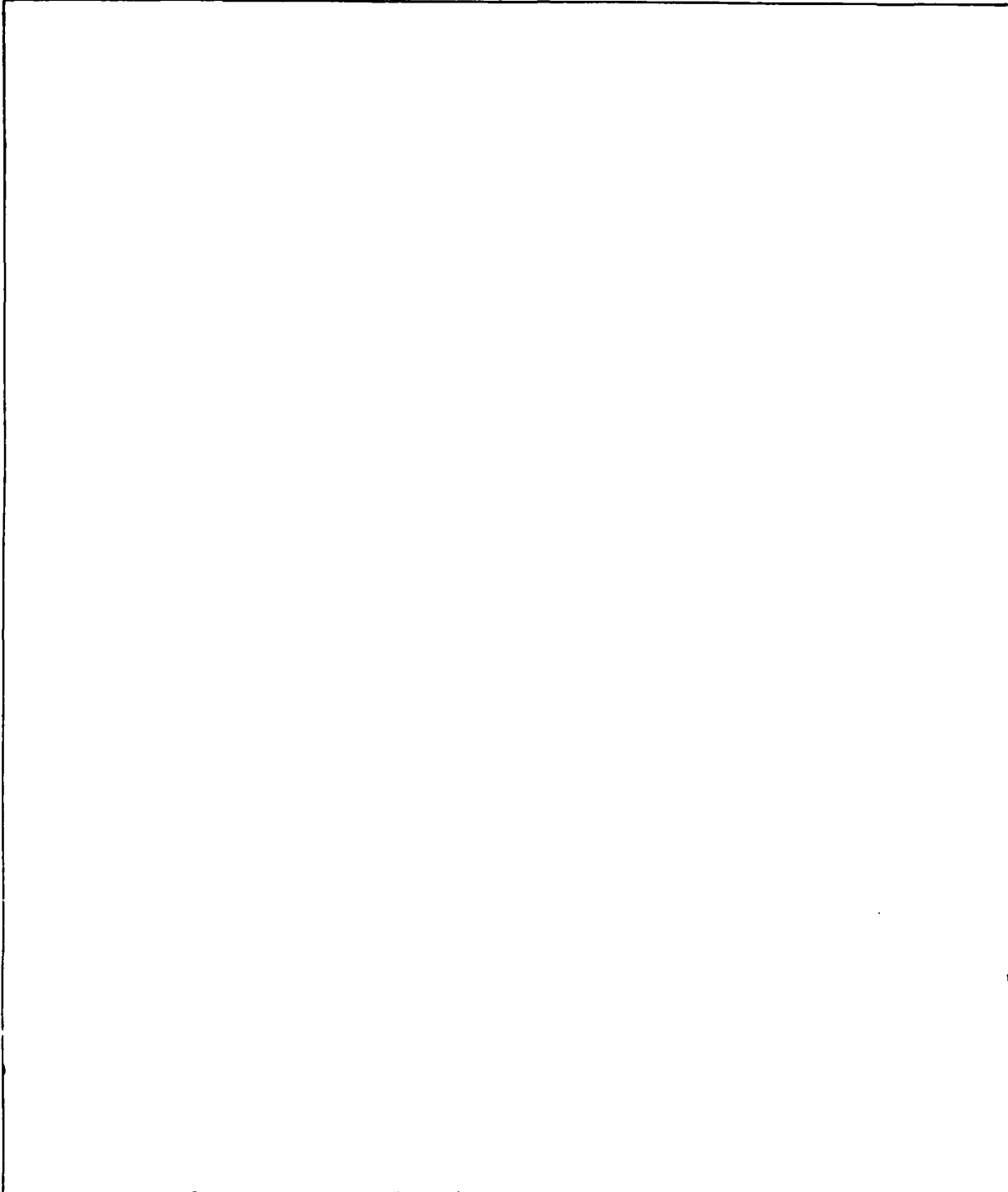
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PREFACE

This investigation was conducted as part of the work unit Behavior of Soil under a Generalized Stress Path, authorized by the US Army Corps of Engineers, under Civil Works Investigation Study (CWIS) Work Unit 32343, "Soil Behavior Under Generalized Stress Paths". This investigation was conducted at the US Army Engineer Waterways Experiment Station (WES) during the period of fiscal years 1985 to 1988.

This report was prepared by Dr. K. C. Valanis, while serving as Professor of Theoretical Mechanics at the University of Cincinnati and as President of Endochronics Inc.; and Dr. J. F. Peters, Geotechnical Laboratory (GL). Dr. Peters served as the Principal Investigator for the project. Research at WES took place under the direct supervision of Mr. G. P. Hale, Chief, Soils Research Center, and the general supervision of Mr. C. L. McAnear, Chief, Soil Mechanics Division, and Dr. W. F. Marcuson III, Chief, GL.

COL Dwayne G. Lee, EN, was Commander and Director of WES during the publication of this report. Dr. Robert W. Whalin was Technical Director.

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THERMODYNAMICS OF FRICTIONAL MATERIALS

A CONSTITUTIVE THEORY OF SOILS WITH DILATANT CAPABILITY

PART I: INTRODUCTION

1. The main task of this work is to use the concepts of endochronic plasticity to derive, from thermodynamic principles, a constitutive equation for soils with dilatant capability. The aim has been to set the theory in a three dimensional framework and endow the resulting constitutive relations with the capability for accounting for the effects of complex deformation histories. The result is a domain of applicability broad enough to capture sufficiently complex phenomena to make the constitutive equations useful in applications to realistic deformation histories encountered in practice.

Overview of Endochronic Plasticity

2. The endochronic theory (Valanis, 1971; Valanis, 1980; Valanis and Read, 1980; Valanis and Lee, 1982; and Valanis, 1986) deals with the plastic response of materials by means of memory integrals expressed in terms of memory kernels which contain the essence of the material behavior. The kernels are invariably fast decaying functions indicating a material memory which fades at a high rate. In the specific case where these kernels are approximated by Dirac delta functions (which decay at an infinite rate) one recovers a theory with yield surfaces. Thus yield surface plasticity is a subset of the endochronic theory.

3. The material memory is specified with respect to an intrinsic time which is the path traversed by the plastic strain in a plastic strain space. If the material is isotropic in its reference configuration the path length is given in terms of one material parameter. See Equation (5), Part II.

4. The application of endochronic plasticity to soils, insofar as work that has appeared in the literature is concerned, has been limited either to one dimensional histories (shear and hydrostatic response) under cyclic and other loading-unloading conditions or to two dimensional histories (shear-volume interaction) under densifying conditions. The question of dilatant behavior had hitherto remained unresolved. The reasoning that led to the present treatment is broadly as follows:

The coupling between deviatoric and hydrostatic behavior that will ultimately lead to dilatant deformation must come from three sources:

- a. The intrinsic time through the parameter k in Equation (5);
- b. The expression for the free energy;
- c. The rate equations for the internal variables.

5. Source a alone, will always give rise to densification as the application of the relevant equations to concrete by Valanis and Read (1986) actually showed. Source b is not physically viable because given a soil with a certain porosity the onset of dilatancy under monotonic shearing is governed by the prevailing hydrostatic stress. If b is to be the source, then a change in the *form* of the free energy must take place upon varying the hydrostatic stress which is not physically plausible. In other words, one would expect the form of the free energy to remain invariant with a change in the hydrostatic stress. The remaining plausible cause is source c and this is the one that we developed in the present work.

Overview of Report

6. In Part II, a simple model is developed to illustrate basic principles. The point of departure is the critical state theory (Schofield and Wroth, 1968) which gives physically valid predictions in stress states near equilibrium. The ideas of critical state are developed beginning on page 6 where a simple constitutive equation is given which exhibits the salient features of densifying and dilatant behavior. In particular, the groundwork is laid with regard to the physical properties of the coupling coefficients that allow for prediction of dilatancy. The simple constitutive equation developed Part II is limited in a number of ways, not the least of which is the fact that it is constrained to obey an isotropic hardening law.

7. An isotropic hardening law may do a reasonable job if no unloadings or sharp kinks are encountered in the deformation path, but otherwise it will give poor results. What does not seem to have been realized in the community of practitioners and other plasticians is that the essence of plastic behavior in materials (almost without exception) insofar as the shear response is concerned is kinematic hardening. In practical thermodynamic terms this means more than one internal variable (in fact a large number or spectrum) is needed if one is to obtain a realistic shear response. This possibility is explored in Part III where the foundations are laid for two distinct approaches, Theory I on page 20 and Theory II on page 28, where two different hydrostatic constitutive equations are derived. Theory II is found to be preferable in the sense that it is not subject to analytical difficulties and approximations and is physically more appealing. In Part IV a specific application is presented using a special form of Theory II.

8. In conclusion, the analytical, physical and thermodynamic foundations are laid for

the derivation of constitutive equations that can account for densifying or dilatant behavior of soils depending on the prevailing density, hydrostatic stress and deformation history. Specifically, it is believed that Equations (152) and (153) have the proper foundation and appropriate analytical form to deal with complex forms of soil behavior. The application of these equations to deformation histories encountered in practice will be the subject of further research.

PART II: FRAMEWORK OF A SIMPLE MODEL

9. This work begins with the simplest model of coupled linear relations between stress and strain-rates. The rates are with respect to the intrinsic time scale z and the coupling is between the hydrostatic and deviatoric strain rates on one hand and corresponding stress states on the other. Specifically, we begin with the Equations (1) and (2) as follows:

$$\mathbf{s} = a_{11} \frac{d\mathbf{e}^p}{dz} + a_{12} \frac{d\epsilon^p}{dz} \quad (1)$$

$$\sigma = a_{21} \cdot \frac{d\mathbf{e}^p}{dz} + a_{22} \frac{d\epsilon^p}{dz} \quad (2)$$

in the usual notation where bold symbols denote tensor quantities and a dot between two tensors denotes their scalar product. The superscript p implies that the strain rates are plastic as defined in terms of the total shear and hydrostatic components as

$$d\mathbf{e}^p = d\mathbf{e} - \frac{1}{2\mu} d\sigma \quad (3)$$

$$d\epsilon^p = d\epsilon - \frac{1}{K} d\sigma \quad (4)$$

where μ and K are the elastic shear and bulk moduli respectively. The intrinsic time measure dz is given by Equation (5)

$$dz^2 = \|d\mathbf{e}^p\|^2 + k^2 |d\epsilon^p|^2 \quad (5)$$

where $\|\cdot\|$ denotes the norm of the tensor and k is the coupling parameter between the deviatoric and hydrostatic deformation. Equations (1) and (2) are the classical thermodynamic rate equations where the "forces" \mathbf{s} and σ on one hand and the "fluxes" $d\mathbf{e}^p/dz$ and $d\epsilon^p/dz$ on the other are related by the matrix of "Onsager" coefficients

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (6)$$

which traditionally have obeyed the symmetry condition

$$a_{rs} = a_{sr} \quad (7)$$

However, as will be discussed presently, the symmetry condition is not essential so long as the positive dissipation inequality (8) is satisfied. Thus, no prior assumptions are made about the symmetry of a_{rs} with respect to r and s and the possibility that $a_{12} \neq a_{21}$ will be considered. Note that both of these are deviatoric because of Equations (1) and (2). In effect a_{12} is deviatoric because s is likewise, and only the deviatoric part of a_{21} plays a role in the inner product on the right hand side of Equation (2). However, the positive rate of dissipation must be observed in which case:

$$s \cdot \frac{de^p}{dz} + \sigma \frac{d\epsilon^p}{dz} > 0 \quad (8)$$

for $\|de^p\| > 0$ and/or $|d\epsilon^p| > 0$. It follows from inequality (8) and Equations (1) and (2) that

$$a_{11} \left\| \frac{de^p}{dz} \right\|^2 + (a_{12} + a_{21}) \cdot \frac{de^p}{dz} \frac{d\epsilon^p}{dz} + a_{22} \left(\frac{d\epsilon^p}{dz} \right)^2 > 0 \quad (9)$$

10. Consider the constraints on the matrix $[A]$ where

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (10)$$

which result from inequality (9). To this end set

$$a_{12} + a_{21} = 2B \quad (11)$$

and let

$$B \cdot \frac{de^p}{dz} = \frac{de^p}{dz} \beta \cos \phi \quad (12)$$

where $\beta = \|B\|$ and $de^p = \|de^p\|$.

11. It follows that inequality (9) will not be violated if the discriminant of the quadratic form Q given by Equation (13).

$$Q = a_{11} \left\| \frac{de^p}{dz} \right\|^2 + 2\beta \cos \phi \left\| \frac{de^p}{dz} \right\| \left| \frac{d\epsilon^p}{dz} \right| + a_{22} \left(\frac{d\epsilon^p}{dz} \right)^2 \quad (13)$$

is negative and a_{11} and a_{22} are positive. Thus the necessary and sufficient condition is

$$a_{11} > 0, \quad a_{22} > 0, \quad \beta^2 \cos^2 \phi - a_{11}a_{22} < 0 \quad (14)$$

for all ϕ . The worst possible case is when $\phi = 0$. Thus the necessary and sufficient condition that inequality (9) be observed is that

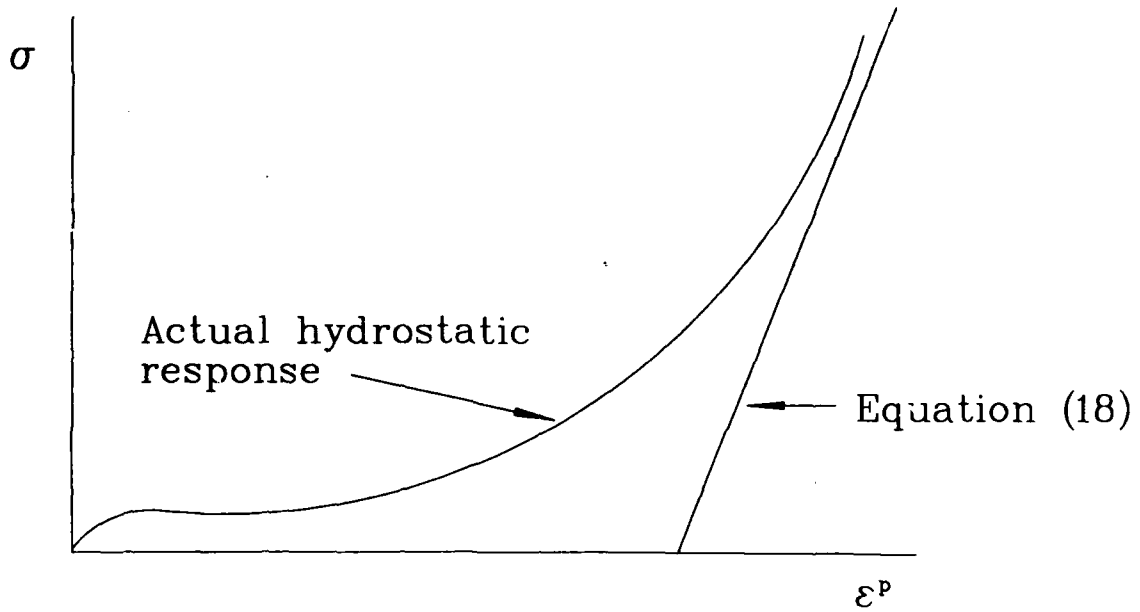


Figure 1: Hydrostatic response depicted by Equation (18).

$$a_{11} > 0, a_{22} > 0, a_{11}a_{22} - \frac{1}{4} \|a_{12} + a_{21}\|^2 > 0 \quad (15)$$

since

$$\beta^2 = \|\mathbf{B}\|^2 = \left\| \frac{1}{2}(a_{12} + a_{21}) \right\|^2 \quad (16)$$

Physical Identification of Matrix [A]

12. In this section connections are established between the proposed simple constitutive theory specified by Equations (1) and (2), and other existing theories which have proved physically sound in describing qualitatively the constitutive response of soils under monotonic loading. One such theory is that of the critical state developed by Roscoe et. al. (1958). The fundamental assumption that underlies this theory is that the rate of plastic work is proportional to the hydrostatic stress. This statement is given analytically in incremental form by Equation (17).

$$\mathbf{s} \cdot d\epsilon^p + \sigma d\epsilon^p = \sigma M d\epsilon^p \quad (17)$$

One observes that under purely hydrostatic conditions Equation (17) leads to the result

$$\sigma d\epsilon^p = 0 \quad (18)$$

which implies that either $\sigma = 0$ or $d\epsilon^p = 0$, that is, either $\sigma = 0$ or the response is elastic. Equation (18), therefore, leads to the hydrostatic response depicted in Figure 1 when the elastic part of the response is linear. Also depicted in that figure is the actual response which qualitatively looks like the critical state model. In presentation of the critical state theory (e.g. Schofield and Wroth, 1968), Equation (17) is usually restricted to the case where $d\epsilon^p \neq 0$. However, one would rather have the response for all loading conditions to be contained within the theory without additional stipulations.¹ Since, otherwise, the critical state theory gives physically sound results, Equations (1) and (2) must contain Equation (17) in some sense. Specifically Equation (2) could, in fact, contain Equation (17) if

$$a_{21} = a_{21}^0 s, \quad a_{22} = a_{22}^0 \sigma \quad (19)$$

where a_{21}^0 and a_{22}^0 are constants. In this specific instance Equation (28) would have the following incremental form:

$$\sigma dz = a_{21}^0 s \cdot d\epsilon^p + a_{22}^0 \sigma d\epsilon^p \quad (20)$$

In the very particular case where $a_{21}^0 = a_{22}^0$ Equation (20) would read

$$\sigma dz = a_{22}^0 (s \cdot d\epsilon^p + \sigma d\epsilon^p) \quad (21)$$

which is a statement to the fact that the increment in plastic work is proportional to the hydrostatic stress, except that dz is not equal to $d\epsilon^p$ as it would be in the case of the critical state theory. However, one would like to retain the greater freedom of Equation (20) rather than bound by the Equation (21). Again observe that under purely hydrostatic conditions and in view of Equations (8) and (20) one finds that

$$\sigma d\epsilon^p (k - a_{22}^0) = 0 \quad (22)$$

Now if $k \neq a_{22}^0$ one again obtains the hydrostatic response in accordance with the critical state theory, given by Equation (18) and depicted in Figure 1. However, there is another choice. By setting

¹The indeterminate nature of the response for purely hydrostatic loading was the motivation of introducing the elliptical surface into the critical state theory by Burland (1965) through the introduction of an alternative form of the right-hand side to Equation (17). Equation (17), which is based on the Taylor energy relationship, typically fits experimental data well and rightly held an axiomatic position in the original theory. Here, the necessity for stipulating $d\epsilon^p \neq 0$ will be removed. The more general problem of indeterminate response under hydrostatic loading will be addressed beginning on page 15.

$$a_{22}^0 = k \quad (23)$$

Equation (22) is satisfied identically so that nothing can be said about either σ or $d\epsilon^p$ and any purely hydrostatic response is admissible (such as the one found in experiment).

13. In view of the above discussion set:

$$a_{12} = a_{12}^0 s \quad (24)$$

$$a_{21} = a_{21}^0 s \quad (25)$$

$$a_{22} = a_{22}^0 \sigma \quad (26)$$

in which case Equations (1) and (2) become

$$s = a_{11} \frac{de^p}{dz} + a_{12}^0 s \frac{d\epsilon^p}{dz} \quad (27)$$

$$\sigma = a_{21} s \cdot \frac{de^p}{dz} + a_{22}^0 \sigma \frac{d\epsilon^p}{dz} \quad (28)$$

Constitutive Response for Case of $a_{12}^0 = 0$

14. This section examines the role of the coefficient a_{12}^0 in Equation (27). It follows from this equation that

$$s = \frac{a_{11} \frac{de^p}{dz}}{1 - a_{12}^0 \frac{d\epsilon^p}{dz}} \quad (29)$$

Whatever the sign of a_{12}^0 a case can be made that the shear response depicted by Equation (29) is unrealistic. For instance in the light of the convention that ϵ^p is positive in compression let a_{12}^0 be positive. One can then envision a situation, where the hydrostatic deformation is densifying, in which case the shear response will become unstable if

$$a_{12}^0 \frac{d\epsilon^p}{dz} \rightarrow 1 \quad (30)$$

Since such possibilities are not physically realistic set $a_{12}^0 = 0$.

Existence of a Yield Surface

15. In this section Equations (27) and (28) are used in conjunction with Equation (23) and the definition of the intrinsic time scale, given by Equations (5) and (8), to derive a yield surface. Substitution of the former into the latter leads to the following result:

$$\sigma^2 - 2a_{11}a_{21}^o\sigma + \|a_{21}^os\|^2 = 0 \quad (31)$$

The geometry of the yield surface will depend on the nature of the functional dependence of a_{11} on σ and z . To examine various forms of its geometry let $\|s\|$ be denoted by q in which event Equation (31) becomes

$$\sigma^2 - 2a_{11}a_{21}^o\sigma + (a_{21}^o)^2q^2 = 0 \quad (32)$$

16. First examine the specific case where

$$a_{11} = a_{11}^o\sigma f(z) \quad (33)$$

In view of Equation (27) and the fact that $a_{12}^o = 0$, Equation (33) implies that the shear stress is directly proportional to the hydrostatic pressure, in accordance with the classical law of friction. In this case, Equation (32) becomes

$$\sigma^2(2a_{11}^oa_{21}^of(z) - 1) = (a_{21}^o)^2q^2 \quad (34)$$

or

$$q = \frac{\sqrt{(2a_{11}^oa_{21}^of(z) - 1)}}{a_{21}^o} \sigma \quad (35)$$

Thus the yield locus becomes a straight line, the slope of which increases as the material hardens, i. e. as $f(z)$ increases with deformation. When $f(z)$ becomes a constant then a "failure line" (surface) is reached.

17. Another case worth investigation is the one where a_{11} is independent of σ . This is not realistic in the case of soils but is of academic interest none-the-less. In this case

$$a_{11} = a_{11}^of(z) \quad (36)$$

and the yield equation given by Equation (32) now becomes

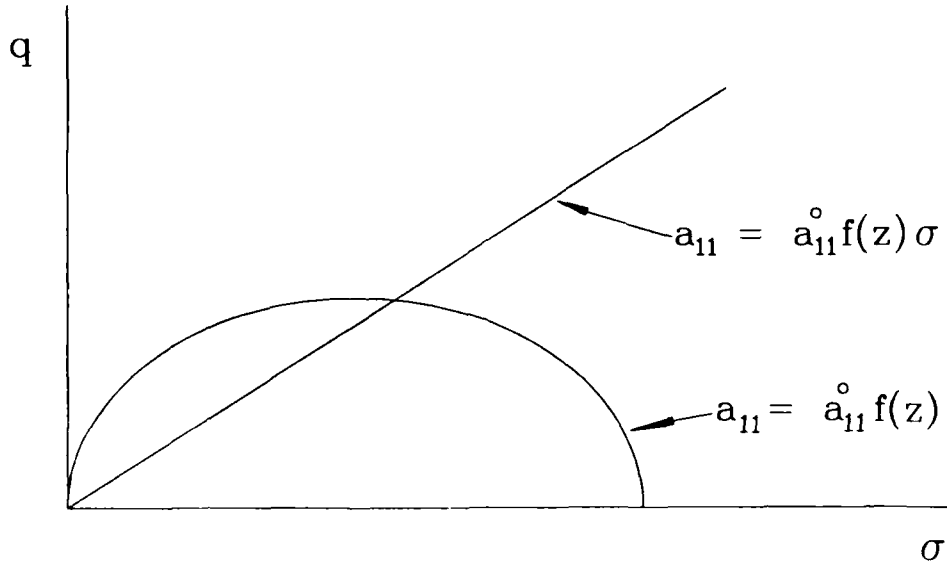


Figure 2: Special types of yield surfaces.

$$\sigma^2 - 2a_{21}^{\circ}a_{11}^{\circ}f(z)\sigma + (a_{21}^{\circ})^2q^2 = 0 \quad (37)$$

or

$$(\sigma - a_{21}^{\circ}a_{11}^{\circ}f(z))^2 + (a_{21}^{\circ})^2q^2 = (a_{21}^{\circ}a_{11}^{\circ}f(z))^2 \quad (38)$$

which is the equation of an ellipse with center $(a_{21}^{\circ}a_{11}^{\circ}f(z), 0)$. These two simple cases are illustrated in Figure 2.

18. In reality the dependence of a_{11} on σ is likely to lie between these two extremes. Thus with reference to Equation (32), a_{11} is likely to be of a form such as:

$$a_{11} = \sigma A(\sigma) a_{11}^{\circ} f(z) \quad (39)$$

where $A(\sigma)$ is decreasing function of σ , $A(0) = 1$, and $a_{11}^{\circ} f(z)$ is a monotonically increasing function of z , becoming constant as $z \rightarrow \infty$. To examine the effect of Equation (39) on the geometry of the yield surface one can choose, typically, $A(\sigma)$ to be the following decaying exponential,

$$A(\sigma) = e^{-a\sigma} \quad (40)$$

in which case Equation (32) becomes

$$\sigma^2(2a_{11}^{\circ}f(z)a_{21}^{\circ}e^{-a\sigma} - 1) = (a_{21}^{\circ})^2q^2 \quad (41)$$

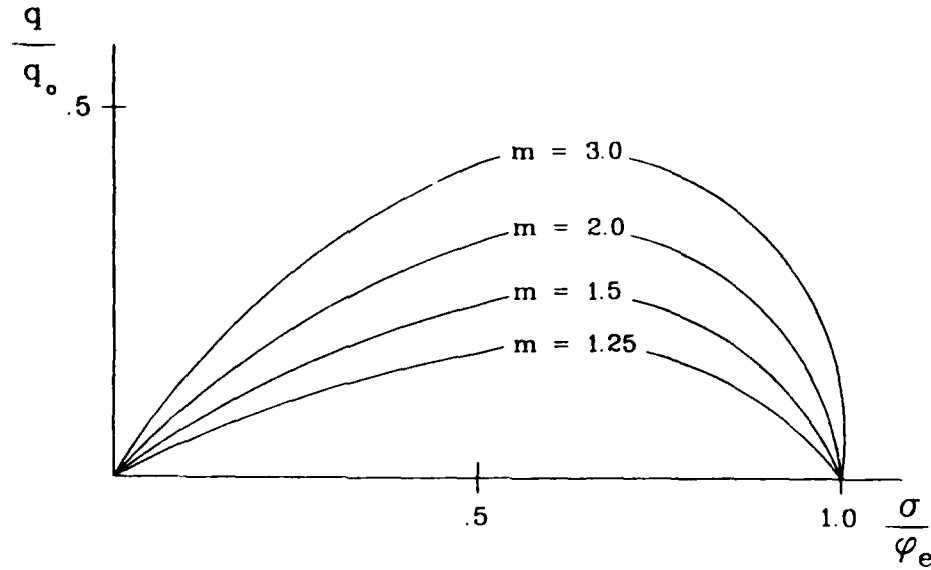


Figure 3: Yield surfaces for various values of m .

Now observe that q is zero at $\sigma = 0$ but also at a limiting hydrostatic stress $\sigma = \phi_e$ such that

$$2a_{11}^0 f(z) a_{21}^0 e^{-a\phi_e} = 1 \quad (42)$$

or

$$a\phi_e = \log m(z) \quad (43)$$

where we have set $2a_{11}^0 a_{21}^0 f(z) = m(z)$. Thus in terms of these parameters, where $m(z)$ is a monotonically increasing function of z with the property

$$\lim_{z \rightarrow \infty} m(z) = m_\infty \quad (44)$$

Equation (41) becomes

$$\frac{q}{q_0} = \frac{\sigma}{\phi_e} \sqrt{m \left(\frac{1}{m} \right)^{\sigma/\phi_e} - 1} \quad (45)$$

where $a_{21}^0 = \phi_e/q_0$. Equation (45) gives a family of yield surfaces each corresponding to an increasing value of $m(z)$. In Figure 3, a subset of these surfaces is shown for various values of $m(z)$. The situation is more complex when a_{11} is also a function of density.

Strain Response Given the Stress History

19. Recall Equations (27) and (28) which, in light of the stipulation that $a_{12}^0 = 0$, become

$$s = a_{11} \frac{de^p}{dz} \quad (46)$$

$$\sigma = a_{21}^0 s \cdot \frac{de^p}{dz} + a_{22}^0 \sigma \frac{d\epsilon^p}{dz} \quad (47)$$

Use of Equation (46) in (47) gives the relation

$$\left(\sigma - \frac{a_{21}^0 q^2}{a_{11}} \right) dz = a_{22}^0 \sigma d\epsilon^p \quad (48)$$

Equation (48) is central in determining whether the volumetric response will be contractive or dilatant. Adopting the convention that σ and ϵ^p are both positive in compression, then $d\epsilon^p > 0$ implies contraction and $d\epsilon^p < 0$ implies dilatancy. Thus if

$$\frac{a_{21}^0}{a_{11}} q^2 > \sigma \quad (49)$$

then the current deformation is dilatant. From Equation (32) it is apparent that σ and q are related since they both have to lie on the yield surface. Specifically, if Equation (39) is substituted into Equation (32) the yield relationship is given by the following:

$$\sigma^2 - 2\sigma^2 A(\sigma) a_{11}^0 f(z) a_{21}^0 + (a_{21}^0)^2 q^2 = 0 \quad (50)$$

or

$$q^2 = \left(\frac{\sigma}{a_{21}^0} \right)^2 [2A(\sigma) a_{11}^0 f(z) a_{21}^0 - 1] \quad (51)$$

Now define quantity Δ such that

$$\Delta = a_{21}^0 a_{11}^0 f(z) A(\sigma) - 1 \quad (52)$$

By substitution of Equation (52) into (51) and combining the result with inequality (49), the inequality can be written as:

$$\frac{2\Delta + 1}{\Delta + 1} > 1 \quad (53)$$

Thus the following situations arise:

$$\Delta > 0 \Rightarrow \text{dilatancy}$$

$$\Delta < 0 \Rightarrow \text{contraction}$$

$$\Delta = 0 \Rightarrow \text{constant-volume deformation.}$$

It is noteworthy that in deriving the above result, Equation (51) was written in the form of Equation (54).

$$2\Delta = \left(a_{21}^0 \frac{q}{\sigma}\right)^2 - 1 \quad (54)$$

Thus the sign of Δ is also determined by Equation (54). In other words, in $q - \sigma$ space there exists a straight line given by the relation

$$a_{21}^0 q = \sigma \quad (55)$$

such that stress points (always on or below the failure envelope) above the straight line give rise to dilatancy but stress points below the line give rise to contraction. Points on the straight line give rise to deformation at constant volume. The situation is illustrated in Figure 4.

Calculation of the Volumetric Response

20. In this section, the volumetric response during shear and its consequences are considered in more detail beginning with an explicit calculation of the intrinsic time by making use of Equations (46) and (47). Substitution of the former in the latter gives the relation

$$\sigma = a_{11}a_{21}^0 \left\| \frac{d\epsilon^p}{dz} \right\|^2 + k\sigma \frac{d\epsilon^p}{dz} \quad (56)$$

where use was made of Equation (23). Use of Equations (39) and (52) gives the additional relation

$$a_{11}a_{21}^0 = \sigma(1 + \Delta) \quad (57)$$

At this point, Equations (56) and (5) combine to give the following quadratic relation for $d\epsilon^p/dz$.

$$(1 + \Delta) \left(k \frac{d\epsilon^p}{dz} \right)^2 - k \frac{d\epsilon^p}{dz} - \Delta = 0 \quad (58)$$

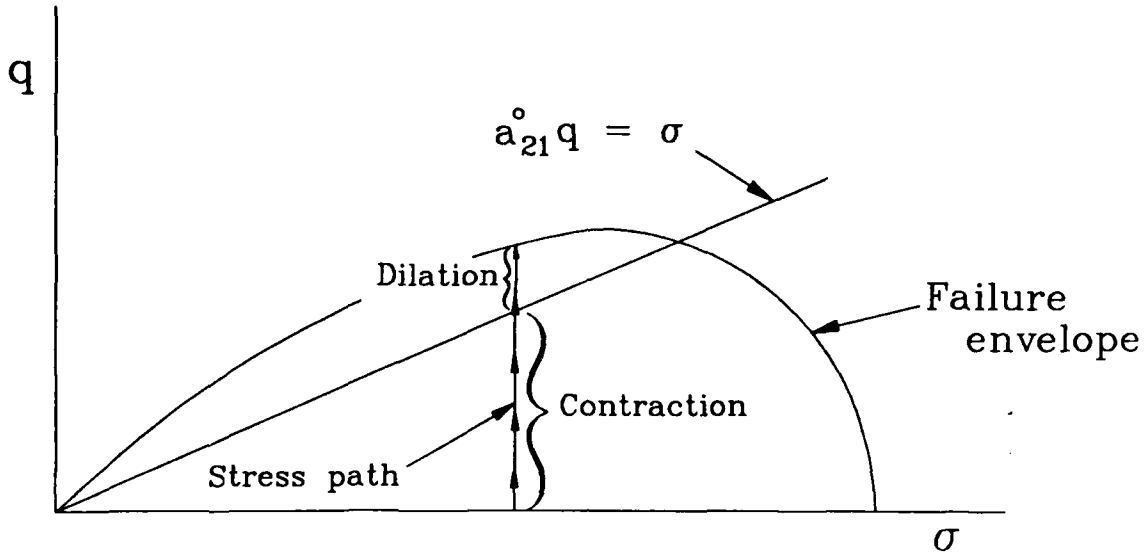


Figure 4: Separation between contractive and dilative states.

which gives two solutions for $k d\epsilon^p/dz$. The one is given by Equation (59)

$$k \frac{d\epsilon^p}{dz} = -\frac{\Delta}{1+\Delta} \quad (59)$$

The second is equal to unity which must be rejected since it gives rise to the consequence that

$$\left\| \frac{d\epsilon^p}{dz} \right\| = 0$$

for all histories (in view of Equation (56)), with the implication that the shear response is always elastic. This, of course is not admissible, except in situations where the deformation is purely hydrostatic. Equation (59) gives rise to the conclusion arrived at earlier that if $\Delta > 0$, then $d\epsilon^p < 0$ and the deformation is dilatant, while if $\Delta < 0$ the material response will be contractive. It follows from Equation (56), in the light of Equation (59) that

$$\left\| \frac{d\epsilon^p}{dz} \right\|^2 = \frac{1+2\Delta}{(1+\Delta)^2} \quad (60)$$

thus

$$dz = \frac{1+\Delta}{\sqrt{1+2\Delta}} \|d\epsilon^p\| \quad (61)$$

Since dz is real we require that the radical should be positive, whereby the following inequality must apply:

$$\Delta > -\frac{1}{2} \quad (62)$$

Recalling Equations (39) and (52) the implication of inequality (62) is that, for $\sigma > 0$ (compression)

$$2a_{11}a_{21}^2 > \sigma \quad (63)$$

Of particular importance is the situation where the equality (64) applies:

$$2a_{11}a_{21}^2 = \sigma \quad (64)$$

in this case and in view of the yield condition given by Equation (31), it follows that $q = 0$. Thus $d\epsilon^p$ is also zero and since $1 + 2\Delta = 0$, dz is indeterminate from Equation (61). Thus the second root of the quadratic Equation (58) must be chosen, i.e., under purely compressive hydrostatic conditions

$$k \frac{d\epsilon^p}{dz} = 1 \quad (65)$$

as it should be in view of Equation (5). Thus, the theory is internally consistent.

Coupling Mechanisms Re-examined

21. This section deals in greater detail with the coupling mechanisms that underlie the deviatoric and hydrostatic stress responses. From a thermodynamic viewpoint the coefficient a_{22} in the rate Equation (2) accounts for the frictional dissipation associated with volumetric plastic deformation. The indeterminate hydrostatic response, given by the simple model under purely hydrostatic conditions, is a consequence of Equation (19) in which a_{22} is proportional to σ . The model thus describes an ideal locking material which displays no plastic volumetric deformation in response to an increase in hydrostatic stress. Although Equation (23) permits the introduction of an independent hydrostatic constitutive equation there is no apparent physical motivation to guide the choice of such an equation.

22. In reality, the coefficient a_{22} in a frictional material will depend very strongly on the prevailing hydrostatic stress because the interparticle frictional forces depend on the normal forces, which are represented in an average sense by σ . In addition, however, a significant part of the dissipation under hydrostatic conditions is due to particle crushing. It is reasonable to suppose that the degree of crushing will increase with the number of contact points associated

with an increase in the density of a soil. These two mechanisms are essentially independent since the number of friction points will not affect the total frictional force (Amonton's Law).

23. Therefore, from a physical standpoint, a_{22} will depend on σ as well as the porosity (or equivalently the density or plastic volumetric strain) and the dependence is likely to be substantially additive. Thus set

$$a_{22} = (1 - \phi_1)\phi_o(n) + \phi_1\sigma \quad (66)$$

where ϕ_o is a function of the porosity n , and ϕ_1 is a constant such at $0 \leq \phi_1 \leq 1$. For purely hydrostatic conditions

$$\sigma = ((1 - \phi_1)\phi_o(n) + \phi_1\sigma) \frac{d\epsilon^p}{dz} \quad (67)$$

which, in view of the definition of intrinsic time, gives the following hydrostatic stress-plastic volumetric strain relation, under monotonic conditions of continuous compression:

$$\sigma = \frac{1 - \phi_1}{k - \phi_1} \phi_o(n) \quad (68)$$

Note that in the case where k is equal to ϕ_1 , the hydrostatic stress becomes infinite. This behavior is that of an ideal locking material discussed previously. However, it is essential that $k > \phi_1$ to ensure a positive (compressive) volumetric strain.

24. It is useful at this point to introduce a porosity-dependent function $\phi_e(n)$ such that

$$\phi_o = \frac{k - \phi_1}{1 - \phi_1} \phi_e(n) \quad (69)$$

The meaning of $\phi_e(n)$ becomes evident in a purely hydrostatic monotonically compressive test whereby Equations (68) and (69) give

$$\sigma = \phi_e(n) \quad (70)$$

Thus, $\phi_e(n)$ is the hydrostatic response function under purely hydrostatic conditions of increasing compressive stress. It is mentioned in passing that Equation (70) allows a connection with Hvorslev's (1937) equivalent pressure concept wherein ϕ_e is identical to the limiting hydrostatic stress introduced in Equation (42). As in Hvorslev's theory the shear strength of the soil becomes a function of the ratio σ/ϕ_e .

25. By substitution of Equation (66) into Equation (2) and use of Equation (5) the yield condition can be written in the form shown below.

$$q^2 + a_{11}^2 \left(1 - \frac{a_{21}^0 q^2}{a_{11} \sigma} \right)^2 \left(\frac{k\sigma}{(1 - \phi_1)\phi_0 + \phi_1\sigma} \right)^2 = a_{11}^2 \quad (71)$$

It can be readily shown that, if $\phi_0 = 0$ and $k = \phi_1$ Equation (32) is recovered. In view of Equation (68), it is possible to make Equation (71) approximate the ideal frictional response of Equation (32) as closely as desired and still retain the hydrostatic response of Equation (68). Specifically, we make use of Equation (69) and define the coupling parameter c as

$$c = \frac{k}{\phi_1} \quad (72)$$

to rewrite the yield relationship as

$$q^2 + a_{11}^2 \left(1 - \frac{a_{21}^0 q^2}{a_{11} \sigma} \right)^2 \left(\frac{c\sigma}{(c - 1)\phi_e + \sigma} \right)^2 = a_{11}^2 \quad (73)$$

Regardless of the magnitude of ϕ_e , Equation (73) may be made as close to Equation (32) as desired by setting c as close to unity as necessary. Thus Equation (73) encompasses the behavior of idealized granular material while eliminating the indeterminacy of the response for purely hydrostatic conditions.

26. It may be shown by direct substitution that, for $q = 0$, Equation (73) reduces to Equation (68). On the other hand, the complete yield equation which is analogous to Equation (45), can be written as

$$\frac{q}{q_0} = \frac{\sigma}{\phi_e} \sqrt{m \left(\frac{1}{m} \right)^{\sigma/\phi_e} - 1} \frac{\sqrt{C}}{B} \quad (74)$$

where

$$C = 1 + \sqrt{1 + 2B^2(1 - B^2) \frac{m^2(1/m)^{2\sigma/\phi_e}}{(m(1/m)^{\sigma/\phi_e} B^2 - 1)^2}} \quad (75)$$

and

$$B = \frac{c\sigma}{(c - 1)\phi_e + \sigma} \quad (76)$$

It is readily verified that Equation (74) is identical to Equation (45) when $c = 1$.

27. Thus far, the parameter c has been treated as a device to formally introduce a hydrostatic yield stress into the theory; in fact, c has an important physical interpretation. By combining Equations (27), (28), (39) and (72) and noting $q_{12}^0 = 0$ the relationship for the rate of volume change is found to be

$$\frac{d\epsilon^p}{dz} = \frac{1 - \frac{a_{21}^2}{a_{11}^2 A(\sigma) f(z)} \left(\frac{q}{\sigma}\right)^2}{\left[(c - \phi_1) \frac{\phi_e}{\sigma} + \phi_1\right]} \quad (77)$$

The effect of $c > 1$ is to reduce the volume change induced by shear-volume coupling at small values of σ . In view of the assumed dependence of ϕ_e on density, the degree that volume change is suppressed by c depends on the compactness of the soil. There is experimental justification for both the reduced shear-normal coupling at low stress and the influence of density as depicted in Equation (77) although quantitative comparison with data remain for future research.

Summary of Simple Model

28. The simple model presented in this part is limited to isotropic hardening. However, several important principles outlined in its development will have application to the more general theory that is presented in the next part:

- a. The coupling between volume change and shear is derived from *both* the definition of endochronic time and the coupling terms in the rate equation.
- b. The indeterminate response to purely hydrostatic loading, typical of models for ideally frictional materials, is removed by introducing a hydrostatic yield stress. Physically, the hydrostatic yield stress is a measure of energy dissipated as a result of particle crushing.
- c. A yield surface was derived from the definition of endochronic time and the rate equations. By making the resistance coefficient proportional to mean stress a model for an ideal Coulomb material is derived. An elliptical yield surface results from the unrealistic assumption that the shear stress at yield is independent of normal stress (as a result of Equations (27) and (36)) even though the rate of energy dissipation is proportional to σ (as a result of Equation (28)).

29. Finally, it is important to point out one further tie to conventional theory of plasticity for frictional materials. Owing to the presence of stress terms in the resistance coefficients $[A]$, the plastic strain increment vectors are not normal to the derived yield surface. Thus, the theory developed is similar to a plasticity theory using a non-associated flow rule. To impose normality within the context of the theory, it would be necessary to propose a purely cohesive mechanism of resistance whereby $[A]$ would not be functions of stress, in which case the frictional nature of the behavior would be lost.

PART III: GENERALIZATION OF THE SIMPLE THEORY

30. The question that poses itself is how to generalize the simple model presented in the previous sections. Though other possibilities may exist, it is apparent that two choices present themselves at the present juncture. One is an *internal variable* generalization of Equations (1) and (2) whereby the plastic strains e^p and ϵ^p are replaced by r internal variables $q^{(r)}$ and $q^{(r)}$ respectively. The r 'th set of the corresponding equations reads:

$$\frac{\partial \Psi_d}{\partial q^{(r)}} + a_{11}^{(r)} \frac{dq^{(r)}}{dz} + a_{12}^{(r)} \frac{dq^{(r)}}{dz} = 0 \quad (78)$$

$$\frac{\partial \Psi_h}{\partial q^{(r)}} + a_{21}^{(r)} \cdot \frac{dq^{(r)}}{dz} + a_{22}^{(r)} \frac{dq^{(r)}}{dz} = 0 \quad (79)$$

and Ψ_d and Ψ_h are given by the following expressions respectively:

$$\Psi_d = \frac{1}{2} \sum_{r=1}^N A_r \|e^p - q^{(r)}\|^2 \quad (80)$$

$$\Psi_h = \frac{1}{2} \sum_{r=1}^N B_r |\epsilon^p - q^{(r)}|^2 \quad (81)$$

This approach will be developed in detail beginning on page 20.

31. The other choice presents itself in the light of the stipulation that $a_{12} = 0$ in Equation (1) and the relation

$$a_{21} = a_{21}^0 s \quad (82)$$

in the light of which Equation (2) becomes:

$$\sigma - a_{21}^0 s \cdot \frac{de^p}{dz} = a_{22} \frac{d\epsilon^p}{dz} \quad (83)$$

32. Equation (83) may now be interpreted as meaning that there are two causes of plastic volumetric strain. One is the hydrostatic stress and the other the rate of deviatoric plastic work. However, from a physical point of view, the deviatoric plastic work is *external* to the hydrostatic process, as far as isotropic materials are concerned at least, and therefore it qualifies as a thermodynamic force of the *first kind* in the sense of Valanis (1983). The concept was discussed in great detail by Valanis in that reference. The thermodynamic formal structure of Equation (83) then becomes

$$\sigma + R = a_{22} \frac{d\epsilon^p}{dz} \quad (84)$$

where R is a thermodynamic (external) force of the "first kind" such that

$$R = -a_{21}^0 s \cdot \frac{de^p}{dz} \quad (85)$$

The generalization of Equation (84) to the case of N internal variables is given by Equations (86) and (87):

$$\sigma = \frac{\partial \Psi_h}{\partial \epsilon^p} \quad (86)$$

$$\frac{\partial \Psi_h}{\partial q^{(r)}} + a_{22}^{(r)} \frac{dq^{(r)}}{dz} = R_r \quad (87)$$

where

$$R_r = -a_{21}^{(r)} s \cdot \frac{de^p}{dz} \quad (88)$$

33. In other words, insofar as the shear-induced volumetric strain is concerned, the effect of the deviatoric plastic work is to apportion itself among the N hydrostatic internal mechanisms, through the coefficients $a_{21}^{(r)}$. This second approach will be developed in greater detail beginning on page 28.

Generalized Theory I

34. Equations (1) and (2) are a particular case of the more general set of equations derived from linear thermodynamics of internal variables. To show this, begin with the relations:

$$s = \frac{\partial \Psi_d}{\partial e^p} \quad (89)$$

$$\sigma = \frac{\partial \Psi_h}{\partial \epsilon^p} \quad (90)$$

where Ψ_d is a quadratic function of e^p and $q^{(r)}$, Ψ_h is a quadratic function of ϵ^p and $q^{(r)}$; $q^{(r)}$ and $q^{(r)}$ are respectively the deviatoric and hydrostatic component of an internal variable $a^{(r)}$ such that

$$a^{(r)} = q^{(r)} + Iq^{(r)} \quad (91)$$

where I is the identity tensor and

$$\Psi(\mathbf{a}^{(r)}, \epsilon^p) = \Psi_d(\mathbf{e}^p, \mathbf{q}^{(r)}) + \Psi_h(\epsilon^p, q^{(r)}) \quad (92)$$

The decompositions into deviatoric and hydrostatic components inherent in Equations (89) to (92) apply only to the case where the material in question is isotropic in its initial state and in the case of Equation (92) which is also quadratic in its arguments. For details see Valanis and Read (1980).

35. The condition of positive internal dissipation is given by inequality (93)

$$\frac{\partial \Psi_d}{\partial \mathbf{q}^{(r)}} \cdot \dot{\mathbf{q}}^{(r)} + \frac{\partial \Psi_h}{\partial q^{(r)}} \dot{q}^{(r)} < 0 \quad (93)$$

whenever

$$\|\dot{\mathbf{q}}^{(r)}\| > 0, |\dot{q}^{(r)}| \geq 0 \quad (94)$$

or

$$|\dot{q}^{(r)}| > 0, \|\dot{\mathbf{q}}^{(r)}\| \geq 0 \quad (95)$$

In the event that $\mathbf{q}^{(r)}$ and $q^{(r)}$ are independent, inequality (93) reduces to the two separate relations

$$\frac{\partial \Psi_d}{\partial \mathbf{q}^{(r)}} \cdot \dot{\mathbf{q}}^{(r)} < 0, \|\dot{\mathbf{q}}^{(r)}\| > 0 \quad (96)$$

and

$$\frac{\partial \Psi_h}{\partial q^{(r)}} \dot{q}^{(r)} < 0, |\dot{q}^{(r)}| > 0 \quad (97)$$

In linear thermodynamics of isotropic materials the deviatoric and hydrostatic evolutions equations are not coupled. This does not preclude shear-hydrostatic coupling which is brought about by the definition of intrinsic time as shown in Valanis and Read (1980 and 1986). When the coupling is brought about only through intrinsic time the material under pressure will always densify when shear is applied. In Part II, it was shown that the presence of the coupling coefficient a_{21} in Equation (2) gave rise to both dilation and contraction, with dilation occurring when the inequality

$$a_{12}^o \|\mathbf{s}\| > \sigma \quad (98)$$

is satisfied (see Equation (49)). Evidently dilatancy cannot occur when $a_{21}^o = 0$.

36. In the spirit of Part II, therefore, and in reference to Equations (1) and (2) express the evolution equations for the internal variables in the form:

$$\frac{\partial \Psi_d}{\partial \mathbf{q}^{(r)}} + a_{11}^{(r)} \frac{d\mathbf{q}^{(r)}}{dz} + a_{12}^{(r)} \frac{dq^{(r)}}{dz} = 0 \quad (99)$$

$$\frac{\partial \Psi_h}{\partial q^{(r)}} + a_{21}^{(r)} \cdot \frac{d\mathbf{q}^{(r)}}{dz} + a_{22}^{(r)} \frac{dq^{(r)}}{dz} = 0 \quad (100)$$

Notable by their presence are the coupling terms $a_{12}^{(r)}$ and $a_{21}^{(r)}$. However in what follows the possibility of $a_{12}^{(r)} = 0$ for all r will be considered, thus abandoning the Onsager symmetry condition. This was discussed in Part II. For the purpose of integrating Equations (99) and (100) we express Ψ_d and Ψ_h in the usual quadratic form:

$$\Psi_d = \frac{1}{2} \sum_{r=1}^N A_r \left\| \mathbf{e}^p - \mathbf{q}^{(r)} \right\|^2 \quad (101)$$

$$\Psi_h = \frac{1}{2} \sum_{r=1}^N B_r \left| \epsilon^p - q^{(r)} \right|^2 \quad (102)$$

and following Part II, write the coefficients a_{11}, a_{22} , and a_{21} in the form:

$$a_{11}^{(r)} = F_d a_{11}^{o(r)} \quad (103)$$

$$a_{22}^{(r)} = F_h a_{22}^{o(r)} \quad (104)$$

and

$$a_{21}^{(r)} = a_{21}^{o(r)} s \quad (105)$$

where F_d is the deviatoric hardening function (in shear) and depends on the hydrostatic stress, F_h is the hydrostatic hardening function which depends on the hydrostatic stress and density (or porosity n). Both F_d and F_h are also likely to depend on z . Equations (99) and (100) in conjunction with Equations (89) and (90) and in light of Equations (101) and (102) give after standard analysis the following constitutive equations

$$s = \int_0^{z_d} \phi_d(z_d - z') \frac{de^p}{dz'} dz' \quad (106)$$

$$\sigma = \int_0^{z_h} \phi_h(z_h - z') \frac{de^p}{dz'} dz' + \sum_{r=1}^N \int_0^{z_h} \Gamma_r(z_h - z') \frac{1}{F_d} s \cdot \left(-\frac{\partial \Psi_d}{\partial q^r} \right) dz' \quad (107)$$

where

$$dz_d = \frac{dz}{F_d} \quad (108)$$

and

$$dz_h = \frac{dz}{kF_h} \quad (109)$$

The kernel functions ϕ_d , ϕ_h , and Γ_r are given in the usual fashion by sums of positive decaying exponential terms according to Equations (110), (111) and (112)

$$\phi_d = \sum_{r=1}^N A_r e^{-\alpha_r z_d} \quad (110)$$

$$\phi_h = \sum_{r=1}^N B_r e^{-\beta_r z_h} \quad (111)$$

$$\Gamma_r = C_r e^{-\beta_r z_h} \quad (112)$$

where

$$\alpha_r = \frac{A_r}{a_{11}^{o(r)}}, \quad \beta_r = \frac{B_r}{a_{22}^{o(r)}}, \quad C_r = B_r \frac{a_{21}^{o(r)}}{a_{11}^{o(r)} a_{22}^{o(r)}} \quad (113)$$

The second integral of Equation (107) arises from the coupling terms in Equation (100).

37. In deriving Equations (106) and (107) it was assumed that all internal forces were zero, that is

$$\left. \frac{\partial \Psi_d}{\partial q^{(r)}} \right|_{z=0} = 0 \quad (114)$$

$$\left. \frac{\partial \Psi_h}{\partial q^{(r)}} \right|_{z=0} = 0 \quad (115)$$

for all r . While it is necessary that $s = 0$ and $\sigma = 0$ at $z = 0$, Equations (114) and (115) need not be satisfied initially. In fact, it is readily shown from Equations (89), (90), (101) and (102) that

$$s = \sum_{r=1}^N -\frac{\partial \Psi_d}{\partial q^{(r)}} \quad (116)$$

$$\sigma = \sum_{r=1}^N -\frac{\partial \Psi_h}{\partial q^{(r)}} \quad (117)$$

Therefore while the sum of the internal forces must be zero in the initial condition, they may be individually non-zero. This is a fact of some importance as it allows for the possibility of an initial apparent shear stress history for a sample subjected to a purely hydrostatic stress history. Such an apparent shear history could result, for example, from hydrostatically stressing an assemblage of particles that have preferred contact orientations created during deposition. This possibility is consistent with the experimental observation by Arthur and Menzies (1972) that the mode of deposition influences subsequent response to loading. The possibility of treating depositionally-induced anisotropy by a non-zero initial state, however, will not be pursued hence but remains a topic for future research. The same remarks apply to the development of Theory II in the section that follows.

38. Equation (107) is too complex for practical applications. To simplify the second integral on the right hand side of this equation, first consider what conditions must apply for the relation

$$-\frac{\partial \Psi_d}{\partial q^r} = k_r s \quad (118)$$

to hold, where k_r are constants. An analysis using Equations (99) and (101) show that this is possible only if

$$\alpha_1 = \alpha_2 = \dots = \alpha_N = \alpha \quad (119)$$

In this event

$$\phi_d = \phi_d^0 e^{-\alpha z_d} \quad (120)$$

that is, the heredity function in shear is given by a single exponential term. While this is not always, if ever, true, it is a reasonable approximation (to ϕ_d) which allows a significant reduction in the complexity of the right hand side of Equation (107). Thus assuming (119) to be *approximately* true then it follows from Equations (99) and (101) that

$$s = -\sum_{r=1}^N \left(\frac{\partial \Psi_d}{\partial q^{(r)}} \right) = \sum_{r=1}^N A_r \int_0^{z_d} e^{-\alpha(z_d - z')} \frac{de^p}{dz'} dz' \quad (121)$$

But

$$-\frac{\partial \Psi_d}{\partial q^{(r)}} = A_r \int_0^{z_d} e^{-\alpha(z_d - z')} \frac{de^p}{dz'} dz' \quad (122)$$

Thus, in view of Equations (118), (121) and (122)

$$k_r = \frac{A_r}{\sum_{r=1}^N A_r} \quad (123)$$

Now use of Equation (118) in Equation (107) gives the following simplified constitutive equation for the hydrostatic response:

$$\sigma = \int_0^{z_h} \phi_h(z_h - z') \frac{de^p}{dz'} dz' + \int_0^{z_h} \Gamma(z_h - z') \frac{\|s\|^2}{F_d} dz' \quad (124)$$

where

$$\Gamma(z_h) = \sum_{r=1}^N k_r C_r e^{-\beta_r z_h} \quad (125)$$

Thus to summarize, the constitutive description of *potentially* dilatant behavior in soils is given by Equations (106) and (124) in terms of two heredity functions ϕ_d and ϕ_h , the coupling kernel function $\Gamma(z_h)$ and the hardening functions F_d and F_h in addition to the elastic shear and bulk stiffness constants μ and K . The remainder of this section considers the conditions where Equation (124) does indeed give rise to dilatant behavior in soils.

39. The dilatant behavior of Equation (124) is studied most easily (and in closed form) in the particular case where both ϕ_h and Γ are given in terms of a single exponential function, that is, when

$$\phi_h = \beta \phi_h^0 e^{-\beta z_h} \quad (126)$$

and

$$\Gamma = \beta \Gamma_0 e^{-\beta z_h} \quad (127)$$

The multiplier β on the right hand side of Equations (126) and (127) is not put there artificially but it serves a useful purpose in that, in the limiting case of very high β , both ϕ_h and Γ tend asymptotically to Dirac delta functions, which are useful approximations to ϕ_h and Γ for non-cohesive materials such as sands.

40. Using Laplace Transforms one may, in the light of Equations (126) and (127), solve Equation (124) in terms of ϵ^p to obtain the following relation:

$$\epsilon^p = \frac{1}{\phi_h^o} \left(\sigma + \int_0^{z_h} \sigma dz' \right) - \frac{\Gamma_o}{\phi_h^o} \int_0^{z_h} \frac{\|s\|^2}{F_d} dz' \quad (128)$$

Now let σ be equal to σ_o , the constant hydrostatic stress at which shearing begins, and let $d\epsilon_d^p$ be the increment in volumetric strain due to shear at constant hydrostatic stress. Then, differentiating Equation (128) we obtain the following relation for $d\epsilon_d^p$:

$$\phi_h^o d\epsilon_d^p = \left(\sigma_o - \Gamma_o \frac{\|s\|^2}{F_d} \right) dz_h \quad (129)$$

where the following relation was used:

$$\epsilon^p = \epsilon_o^p + \epsilon_d^p \quad (130)$$

ϵ_o^p being the plastic volumetric strain induced by the initial hydrostatic stress σ_o . Thus in this more general approach the criterion for dilatant behavior is given by inequality (131):

$$\|s\|^2 > \frac{F_d}{\Gamma_o} \sigma_o \quad (131)$$

41. The case of a cylindrical triaxial test at constant lateral pressure is more complex since σ does not remain constant during the test. Thus upon differentiation of Equation (128) one does not obtain Equation (129).

42. Given that for cohesionless soils F_d is substantially proportional to σ , then in this particular case and for $\sigma > 0$ (σ positive in compression):

$$F_d = F_d^o \sigma \quad (132)$$

In view of Equation (132) the criterion for dilatancy in non-cohesive soils is given by inequality (133)

$$\|s\| > \sqrt{\frac{F_d^o}{\Gamma_o}} \sigma_o \quad (133)$$

Note that condition (131) and its corollary (133) are identical in form to condition (49) and (54) which was obtained in the case of the simpler theory of Part II.

43. In shear experiments on soils under constant hydrostatic stress what is actually measured is the resulting volumetric (plastic) strain as a function of the shear strain (or shear stress). Thus what one obtains eventually is a relation between the shear induced plastic volumetric strain and the applied plastic shear strain. To obtain such a theoretical relation from Equation (129) use Equations (5) and (109) to obtain the following result on the basis that $de^p > 0$:

$$k \frac{d\epsilon_d^p}{de^p} = \frac{\sigma_o - \Gamma_o \frac{\|s\|^2}{F_d}}{\sqrt{(F_h \phi_h^o)^2 - \left(\sigma_o - \Gamma_o \frac{\|s\|^2}{F_d}\right)^2}} \quad (134)$$

For soils the hardening function F_h is adequately represented by the form

$$F_h = (1 - \phi_h^1) e^{\beta e^p} + \frac{\phi_h^1}{\phi_h^o} \sigma \quad (135)$$

where β and ϕ_h^1 are material properties and ϕ_h^o has the same definition as when introduced before.

44. Equation (134) may be integrated numerically to determine ϵ_d^p as a function of e^p provided that the material constants $\Gamma_o, \phi_h^o, F_d, \beta, \phi_h^1$ and k are known. Actually five material constants are necessary to determine theoretically the shear-induced volumetric response since Equations (132) and (134) dictate that Γ_o and F_d appear in the ratio Γ_o/F_d .

Further Simplification of Theory I

45. The hydrostatic stress-strain curve of soil (obtained under monotonic conditions of increasing hydrostatic stress) consists of an initially convex part which quickly gives way to a concave part which prevails over the remainder of the curve. The concave part of the stress-strain curve is obtained readily by setting $\phi_h(z_h)$ proportional to a Dirac delta function, i.e., by setting in Equations (126) and (127) $\beta = \infty$. In this event

$$\phi_h = \phi_h^o \delta(z_h) \quad (136)$$

$$\Gamma = \Gamma_o \delta(z_h) \quad (137)$$

46. With reference to the purely hydrostatic response ($s = 0$) and in view of Equations (5), (107), (124), and (135), it follows that:

$$\sigma = \phi_h^o e^{\beta \epsilon^p} \quad (138)$$

an expression which agrees remarkably well with experiment in the concave region of the hydrostatic stress-strain curve. Thus

$$\sigma_o = \phi_h^o e^{\beta \epsilon_o^p} \quad (139)$$

Thus in view of Equations (132), (135) and (139) Equation (134) becomes:

$$k \frac{d\epsilon_d^p}{d\epsilon^p} = \frac{1}{D} \left[1 - \frac{\Gamma_o}{F_d^o} \left(\frac{\|s\|}{\sigma_o} \right)^2 \right] \quad (140)$$

where

$$D = \sqrt{\left((1 - \phi_h^1) e^{\beta \epsilon_d^p} + \phi_h^1 \right)^2 - \left(1 - \frac{\Gamma_o}{F_d^o} \left(\frac{\|s\|}{\sigma_o} \right)^2 \right)} \quad (141)$$

This equation may now be integrated numerically provided that the constants k , β , ϕ_o^1 and Γ_o/F_d^o are known.

Generalized Theory II

47. In this section the position is considered that whereas the deviatoric plastic work is a cause of hydrostatic plastic strain, it is external to the hydrostatic process in the sense that it is not a hydrostatic mechanism. Therefore, it qualifies as a thermodynamic internal force of the first kind in the sense of Valanis (1983). In the presence of such forces the thermodynamic equations appropriate to the rigid plastic solid that represents the plastic behavior of the soil are the following:

$$\Psi_d = \frac{1}{2} \sum_{r=1}^N A_r \|e^p - q^{(r)}\|^2 \quad (142)$$

$$\Psi_h = \frac{1}{2} \sum_{r=1}^N B_r |\epsilon^p - q^{(r)}|^2 \quad (143)$$

$$s = \frac{\partial \Psi_d}{\partial e^p} \quad (144)$$

$$\sigma = \frac{\partial \Psi_h}{\partial \epsilon^p} \quad (145)$$

$$\frac{\partial \Psi_d}{\partial q^{(r)}} + a_{11}^{(r)} \frac{dq^{(r)}}{dz} = 0 \quad (146)$$

$$\frac{\partial \Psi_h}{\partial q^{(r)}} + a_{22}^{(r)} \frac{dq^{(r)}}{dz} = R_r \quad (147)$$

where

$$R_r = -a_{21}^{(r)} \mathbf{s} \cdot \frac{d\mathbf{e}^p}{dz} \quad (148)$$

In addition, the resistance coefficients $a_{11}^{(r)}$ and $a_{22}^{(r)}$ are not constant but are related to the hardening functions F_d and F_h respectively by the relations

$$a_{11}^{(r)} = F_d a_{11}^{o(r)}, \quad a_{22}^{(r)} = F_h a_{22}^{o(r)} \quad (149)$$

where

$$F_d = F_d(\sigma, z), \quad F_h = F_h(\sigma, \epsilon^p) \quad (150)$$

48. The dependence of F_d on z is weak so that for deformations other than cyclic histories z plays a minor role in F_d and may be ignored. Also the dependence of F_h on σ and ϵ^p is additive, a fact that was discussed at some length on page 15. A straightforward analysis using Equations (142) to (148) in the light of the initial conditions

$$\mathbf{q}^{(r)}(0) = \mathbf{0}, \quad q^{(r)}(0) = 0 \quad (151)$$

gives rise to the following set of two constitutive equations:

$$\mathbf{s} = \int_0^{z_d} \phi_d(z_d - z') \frac{d\mathbf{e}^p}{dz'} dz' \quad (152)$$

$$\sigma = \int_0^{z_h} \phi_h(z_h - z') \frac{d\epsilon^p}{dz'} dz' + \int_0^{z_h} \Gamma(z_h - z') \mathbf{s} \cdot \frac{d\mathbf{e}^p}{dz'} dz' \quad (153)$$

where

$$dz_d = \frac{dz}{F_d} \quad (154)$$

$$dz_h = \frac{dz}{F_h} \quad (155)$$

$$\phi_d = \sum_{r=1}^N A_r e^{-\alpha_r z_d} \quad (156)$$

$$\phi_h = \sum_{r=1}^N B_r e^{-\beta_r z_h} \quad (157)$$

$$\Gamma = \sum_{r=1}^N \beta_r \frac{a_{21}^{(r)}}{F_h} e^{-\beta_r z_h} \quad (158)$$

$$\alpha_r = \frac{A_r}{a_{11}^{(r)}} \quad (159)$$

$$\beta_r = \frac{B_r}{a_{22}^{(r)}} \quad (160)$$

49. The determination of the form of the functions ϕ_d, ϕ_h and Γ from appropriate experiments is a subject for future research. Some simplifications do occur when the effect of the deviatoric plastic work rate is distributed uniformly among the hydrostatic mechanisms, in which case

$$a_{21}^{o(r)} = a_{21}^o \quad (\text{for all } r) \quad (161)$$

50. In this event, and in view of Equations (157) and (158)

$$\Gamma(z_h) = a_{21}^o \phi_h(z_h) \quad (162)$$

so that once $\phi_h(z_h)$ is known, from a simple monotonic hydrostatic test, then $\Gamma(z_h)$ is also determined to within a multiplicative constant.

Simple Example of Theory II

51. As an illustration, the theory is applied to the simple case of a cylindrical triaxial test when the soil is sheared under constant hydrostatic stress. Specifically the soil is stressed initially under purely hydrostatic conditions until σ is equal to σ_o . With σ held at this value the soil is then compressed axially and the shear stress, the shear strain and the volumetric strain are measured. Also it is assumed that σ_o is sufficiently large to lie on the concave part of the hydrostatic stress-strain curve, so that $\phi_h(z_h)$ can then be represented by delta function. See Valanis and Read (1980).

52. The mathematical consequences of this representation are of interest in view of Equations (157) and (160)

$$\phi_h = \sum_{r=1}^N a_{22}^{o(r)} \beta_r e^{-\beta_r z_h} \quad (163)$$

The limiting process by which ϕ_h tends to a delta function is the one whereby $\beta_r \rightarrow \infty$, in which event

$$\phi_h = \delta(z_h) \sum_{r=1}^N a_{22}^{o(r)} = \phi^o \delta(z_h) \quad (164)$$

Of importance is the fact that when ϕ_h tends to a delta function, so does $\Gamma(z_h)$ by virtue of Equation (158). In effect

$$\lim_{\beta_r \rightarrow \infty} \Gamma(z_h) = \frac{1}{F_h} \sum_{r=1}^N a_{21}^{o(r)} \delta(z_h) \quad (165)$$

In this event

$$\Gamma(z_h) = \Gamma_o \delta(z_h) / F_h \quad (166)$$

where

$$\Gamma_o = \sum_{r=1}^N a_{21}^{o(r)} \quad (167)$$

The constitutive Equation (153) now reduces to the simple form

$$\sigma = \phi_o \frac{d\epsilon^p}{dz_h} + \Gamma_o s \cdot \frac{de^p}{dz} \quad (168)$$

where Equation (155) was also used. This last equation is used to investigate the dilatant behavior of soils under constant hydrostatic stress.

53. Following the deliberations on page 18 we set

$$F_h = (1 - \phi_h^1) e^{\beta \epsilon^p} + \frac{\phi_h^1}{\phi_h^o} \sigma \quad (169)$$

so that Equation (168) becomes

$$\sigma = k((1 - \phi_h^1) \phi_h^o e^{\beta \epsilon^p} + \phi_h^1 \sigma) \frac{d\epsilon^p}{dz} + \Gamma_o s \cdot \frac{de^p}{dz} \quad (170)$$

An analysis of the cylindrical triaxial test shows that if one sets

$$s = \sqrt{\frac{2}{3}} (\sigma_1 - \sigma_3) \quad (171)$$

$$e^p = \sqrt{\frac{2}{3}}(\epsilon_1^p - \epsilon_3^p) \quad (172)$$

where "1" denotes the axial direction and "3" one of the lateral directions, then

$$de^p \cdot de^p = (de^p)^2 \quad (173)$$

$$s \cdot de^p = s de^p \quad (174)$$

Thus the equation that defines the intrinsic time z , on one hand, and Equation (170) on the other become

$$dz^2 = (de^p)^2 + k^2 (d\epsilon^p)^2 \quad (175)$$

$$\sigma = k((1 - \phi_h^1)\phi_h^0 e^{\beta\epsilon^p} + \phi_h^1 \sigma) \frac{d\epsilon^p}{dz} + \Gamma_0 s \frac{de^p}{dz} \quad (176)$$

The ultimate goal is to solve Equations (175) and (176) simultaneously and thus obtain a relation between ϵ^p and e^p .

54. To this end note that under purely hydrostatic conditions ($s = 0, e^p = 0$) for monotonic loading

$$\sigma_0 = \phi_h^0 e^{\beta\epsilon_0^p} \quad (177)$$

for all σ_0 . Therefore define, as before, a deviatorically induced hydrostatic strain ϵ_d^p such that

$$\epsilon_d^p = \epsilon^p - \epsilon_0^p \quad (178)$$

Equations (175) and (176) may now be solved simultaneously under conditions of monotonically increasing e^p to give the following result:¹

$$k \frac{d\epsilon_d^p}{de^p} = \frac{1 - \Gamma_0^2 s^{*2}}{h \Gamma_0 s^* + \sqrt{h^2 + \Gamma_0^2 s^{*2} - 1}} \quad (179)$$

where

$$h = (1 - \phi_h^1) e^{\beta\epsilon_d^p} + \phi_h^1 \quad (180)$$

¹Equation (179) corresponds to the case where s is increasing. A second solution exists for Equations (175) and (176) which corresponds to unloading. The solution procedure leading to Equation (179) is discussed in detail in Appendix A.

and

$$s^* = \frac{s}{\sigma_0} \quad (181)$$

Comparison of Theories

55. The theories presented in the two previous sections differ as a result of the interpretation of the coupling terms in Equations (78) and (79). Theory I results from assuming that coupling occurs for each mechanism whereas Theory II results from assuming that the coupling occurs through some averaging mechanism represented by the external force R . Theory II is quite amenable to simplification without loss of physical relevance whereas Theory I leads to relationships that are virtually intractable except for a limited physically unrealistic case in which the shear response is modeled by one internal variable. It is worth considering, therefore, how the theories differ on physical grounds.

56. Both theories predict a contractive-dilatant response for monotonic loading, while for unloading some important differences arise. Consider an experiment whereby a cylindrical specimen is first loaded axially until dilation is observed then unloaded along the same path. In accordance with Equation (129) Theory I predicts that the volumetric strain upon unloading would be dilative and would remain so provided inequality (131) was satisfied. This result is, in fact, easily extended to the general case described by Equation (107) by noting that the coupling described by the second integral is a function of the internal state and independent of the loading direction.

57. By contrast Theory II predicts that the sign of the volumetric strain increment caused by coupling is controlled by the product $s \cdot de^p$ as seen from either Equation (153) or (170). Thus, Theory II predicts that upon unloading the volumetric rate should change from dilation to contraction (see Appendix A). One of the most important features of the response of frictional materials to cyclic loading is their tendency to densify upon load reversal. Therefore, Theory I appears to be physically inadmissible on this particular basis.

Yield Surface

58. A feature of the simple theory presented in Part II was the existence of a yield surface which provided a means for comparison with models developed from the theory of plasticity. For multiple internal variable formulations, such as those presented in this Part, the boundary in stress space between elastic and plastic response is not necessarily marked by a yield surface. As

shown by Valanis (1980) for metals, a yield surface is a mathematical consequence of representing at least one of the exponential terms of the memory kernel with a delta function. The situation for the models presented here is similar as will be illustrated for Theory II. To begin, the first terms in the series given in Equations (156) to (158) are approximated by delta functions or alternatively by letting A_1, α_1, B_1 , and β_1 become infinite while keeping the ratios A_1/α_1 and B_1/β_1 finite. In this case the integrals in Equations (152) and (153) become Equations (182) and (183):

$$\mathbf{s} = \phi_d^o F_d \frac{d\epsilon^p}{dz} + \int_0^{z_d} \phi'_d(z_d - z') \frac{d\epsilon^p}{dz'} dz' \quad (182)$$

$$\sigma = \phi_d^o \frac{d\epsilon^p}{dz} + \Gamma_o F_h \mathbf{s} \cdot \frac{d\epsilon^p}{dz} + \int_0^{z_h} \phi'_h(z_h - z') \frac{d\epsilon^p}{dz'} dz' + \int_0^{z_h} \Gamma'(z_h - z') \mathbf{s} \cdot \frac{d\epsilon^p}{dz'} dz' \quad (183)$$

which give rise to a theory with a yield surface.

59. Equations (182) and (183) may be used to obtain explicit relationships for the plastic strain rates in terms of the stresses and history integrals. Relationships analogous to Equations (46) and (47) (with $\alpha_{12}^o = 0$) can be now derived as:

$$\mathbf{s} - \mathbf{Q}_s = \phi_d^o F_d \frac{d\epsilon^p}{dz} \quad (184)$$

and

$$\sigma - Q_\sigma = \phi_d^o \frac{d\epsilon^p}{dz} + \Gamma_o F_h \mathbf{s} \cdot \frac{d\epsilon^p}{dz} \quad (185)$$

where

$$\mathbf{Q}_s = \int_0^{z_d} \phi'_d(z_d - z') \frac{d\epsilon^p}{dz'} dz' \quad (186)$$

and

$$Q_\sigma = \int_0^{z_h} \phi'_h(z_h - z') \frac{d\epsilon^p}{dz'} dz' + \int_0^{z_h} \Gamma'(z_h - z') \mathbf{s} \cdot \frac{d\epsilon^p}{dz'} dz' \quad (187)$$

60. Through an analysis similar to that on page 15 whereby the plastic strain rates are substituted into the expression for endochronic time the following yield condition is obtained:

$$\|\mathbf{s} - \mathbf{Q}_s\|^2 + \left(\frac{\phi_d^o F_d}{k \phi_h^o F_h} \right)^2 \left((\sigma - Q_\sigma) - \frac{\Gamma_o}{\phi_d^o F_d} \mathbf{s} \cdot (\mathbf{s} - \mathbf{Q}_s) \right)^2 - \phi_d^{o2} F_d^2 = 0 \quad (188)$$

61. Equation (188) describes a yield behavior that includes both kinematic and isotropic hardening. If F_d and F_h are constant the behavior is purely kinematic hardening. In the case

of an incompressible material where $\phi_h^o F_h \rightarrow \infty$ the deviatoric projection is circular as for the case of metals derived by Valanis(1980). For compressible materials with $\Gamma_o \neq 0$ the projection is non-circular as a result of the product $\mathbf{s} \cdot (\mathbf{s} - \mathbf{Q}_s)$.

62. This is a historical development of some relevance. Note that the constitutive relations (182) and (183) in conjunction with the yield equation (188) contain a kinematic-cum-isotropic hardening rule in the co-joint hydrostatic-deviatoric stress space. This comes about naturally through the vehicle of thermodynamics which has not been available to treatments that *begin* with the concept of the yield surfaces.

PART IV: EXAMPLE APPLICATION TO SAND

63. In this part the theory described on page 28 will be used to model the response of Sacramento River sand in the triaxial compression test. This test is performed by increasing the axial stress σ_1 while keeping the lateral stress σ_3 constant. Thus the average stress σ increases during the test which complicates the analysis of the constitutive equations. Therefore, an important goal of the analysis in this Part is to demonstrate how the theory can be applied to the more common triaxial test, an essential step in use of the model for practical applications. The data used for the analysis is taken from Lee and Seed (1967). These data are presented in Figures 5 and 6.

Hydrostatic Response

64. The hydrostatic response is described by the one-term approximation presented by Equation (168). To simplify the determination of parameters a form of F_h is used to yield a relationship equivalent to Equation (169) given by¹:

$$\phi_o F_h = (1 - \phi_1) \phi_e(n) + \phi_1 \sigma \quad (189)$$

Note that ϕ_1 and $\phi_e(n)$ were previously defined with regard to the simple model described in Part II (see page 15) where $0 \leq \phi_1 \leq 1$ and $\phi_e(n)$ is a density-dependent hydrostatic limit stress. Also note that because ϕ_o always appears as a multiplier to F_h , we can set $\phi_o = 1$ without loss of generality. For purely hydrostatic loading Equation (168) reduces to:

$$\sigma = \phi_e(n) \quad (190)$$

65. For consistency with the data presentation in Figure 5 the specific volume will be used which is related to the porosity by:

$$v = \frac{1}{1 - n} \quad (191)$$

Also, for purposes of computation it is desirable to express $\sigma_o(v)$ in terms of plastic hydrostatic strain. The change in hydrostatic strain can be related to the change in void ratio using the well known relationship:

¹Note that these relationships differ from those in Equations (66) to (70) because of the appearance of k in the definition of dz_h .

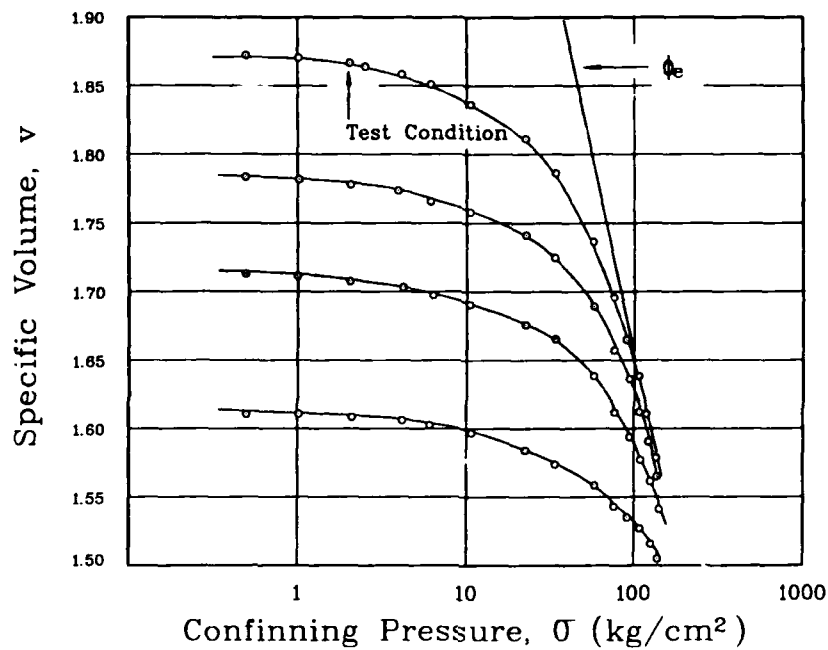


Figure 5: Pressure-specific volume curves for Sacramento River sand at four initial densities

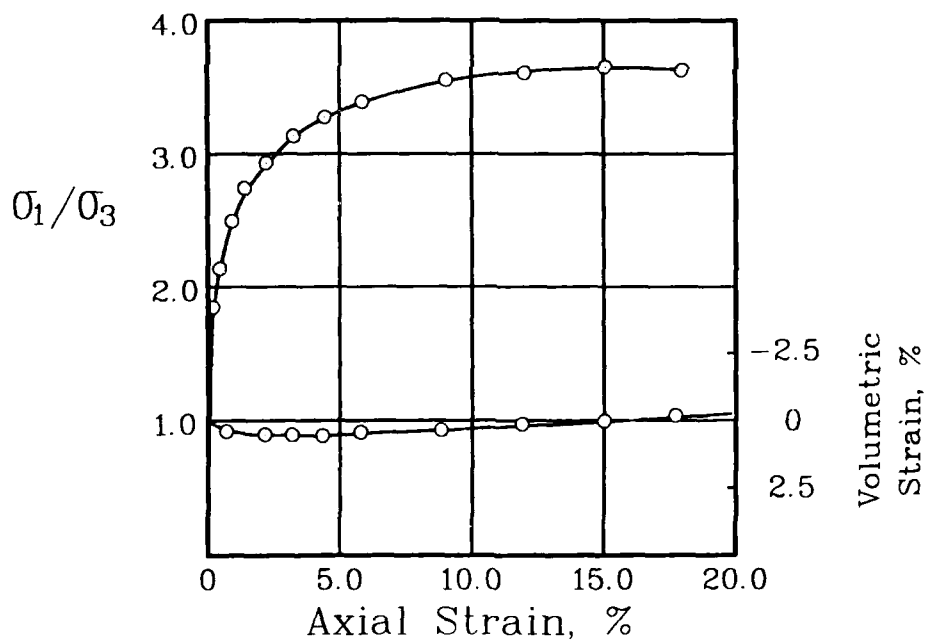


Figure 6: Stress-strain-volume change data for Sacramento River sand with $v_0 = 1.87$ and $\sigma_3 = 2 \text{ kg/cm}^2$

$$\frac{d\epsilon^p}{dv} = -\frac{1}{v} \quad (192)$$

It will be assumed that the initial unstressed state of the sand corresponds to the loosest state. The limiting pressure, ϕ_e can be expressed as a function of specific volume as

$$\phi_e = 20e^{8.61(1-v/v_o)} \quad (193)$$

where v_o is the loosest state obtainable by the soil. This state is taken as the so-called maximum void ratio (e_{maz}) which defines the loosest packing that can be achieved by a standardized test. Lee and Seed reported e_{maz} for Sacramento River sand to be 1.003. Thus $v_o = 2.003$.

66. Equation (193) implies that the initial state does not depend on details of hydrostatic loading history. For example, a loose sand and dense sand, if loaded to sufficiently high stress, can be brought to the same v , even though the two specimens have significantly different stress histories; as far as mechanical response to subsequent loading is concerned, the two samples would be identical. For purely hydrostatic histories this simplistic interpretation of the sample behavior can be justified based on the use of Equation (166) and indeed the assumed equivalence between plastic strain and void ratio is implied in many soil mechanics models based on normalized behavior (e. g. Ladd and Foott, 1974). In practice this approach may introduce significant error because it ignores the apparent shear history introduced through sample densification. The role of sample preparation in the context of the initial condition was discussed in Part III, page 23. Nevertheless, it will be assumed here that all specimens are initially at a state that can be described by an equivalent purely hydrostatic loading history that is specified by the scalar v .

Shear-Volume Coupling

67. Evaluation of the coupling between shear and volume change begins with Equation (168) which can be written for the triaxial compression test conditions through substitution of Equations (109) and (171) as:

$$\sigma = kF_h \frac{d\epsilon^p}{dz} + \Gamma_o s \frac{de^p}{dz} \quad (194)$$

By combination of Equations (175) into (194) the following relationship for rate of volume change is derived in Appendix A:

$$k \frac{d\epsilon_d^p}{de^p} = \frac{1 - \Gamma_o^2 \frac{s^2}{\sigma^2}}{\Gamma_o \frac{F_h s}{\sigma^2} + \sqrt{\frac{F_h^2}{\sigma^2} + \Gamma_o^2 \frac{s^2}{\sigma^2} - 1}} \quad (195)$$

Equation (195) is valid for increasing s and in contrast to Equation (179) is not restricted to shear histories with constant hydrostatic stress. Note that the condition for dilatancy is identical to that derived on page 28 for constant σ .

68. The two constants Γ_o and k can be determined from the slope of the a plot of ϵ^p versus e^p . The constant Γ_o is the ratio s/σ at the point in the test where $d\epsilon^p = 0$ (initiation of dilation). The coupling constant k is computed by substitution of measured dilatancy rates $d\epsilon^p/de^p$ into Equation (195). From Figure 6, $d\epsilon^p/de^p = 0$ at $s/\sigma = 1.05$ which gives $\Gamma_o = 0.95$. At $s/\sigma = 1.14$, $d\epsilon^p/de^p = -0.056$ and $F_h/\sigma = 2.97$. By trial computation $\phi_1 = .8$ was found to give a good fit to the data. Substitution of these values into Equation (195) gives $k = 0.492$.

Shear Response

69. The shear response for the triaxial compression test, obtained by substitution of Equations (171) and (172) into Equation (152), is given as follows:

$$s = \int_0^{z_d} \phi_d(z_d - z') \frac{de^p}{dz'} dz' \quad (196)$$

The above equation can be written as:

$$s = \int_0^{z_d} \phi_d(z_d - z') G(z') dz' \quad (197)$$

where

$$G(z_d) = F_d(z_d) \frac{de^p(z_d)}{dz} \quad (198)$$

Owing to the small values of k and noting $d\epsilon^p/de^p \leq 1$ it is readily determined from Equation (175) that $de^p/dz \approx 1$. Therefore, evaluation of $G(z_d)$ amounts to determining F_d .

70. Equation (198) is complicated by the fact that $G(z_d)$ is proportional to σ which, in contrast to the simple examples of Part III, varies over the history of the test. A simpler relationship can be derived by noting that the history of σ is tied to the history of s through the following linear relationship:

$$\sigma = \sigma_o + bs \quad (199)$$

where σ_o is the hydrostatic stress before shearing and b is a constant equal to zero for constant hydrostatic stress and .408... in the case of constant lateral stress. Upon substitution of Equation (199) into Equation (197) a linear integral equation in s is obtained which can be solved through a straight-forward application of Laplace transforms once the kernel function is defined. For the example considered here the kernel function will be expressed as a delta function and one exponential term given by Equation (200):

$$\phi_d(z_d) = \phi_o^d \delta(z_d) + A_1 e^{-\alpha_1 z_d} \quad (200)$$

which, upon substitution into Equation (197) gives:

$$s = \sigma_o \left(\phi_o^d + A_1 \int_0^{z_d} e^{z_s - z'} dz' \right) + b \left(\phi_o^d s + A_1 \int_0^{z_d} e^{z_s - z'} s dz' \right) \quad (201)$$

Solving for s gives:

$$s = s_\infty + (s_o - s_\infty) e^{-\alpha z_d} \quad (202)$$

where

$$s_o = \sigma_o \frac{\phi_o^d}{1 - b\phi_o^d} \quad (203)$$

$$s_\infty = (\phi_o^d \alpha_1 + A_1) \frac{\sigma_o}{\alpha(1 - b\phi_o^d)} \quad (204)$$

and

$$\alpha = \alpha_1 - \frac{bA_1}{1 - bA_o} \quad (205)$$

71. Based on the peak stress value shown in Figure 6, $s_\infty = 4.28 \text{ kg/cm}^2$. The two constants $s_o - s_\infty$, and α can be obtained from the test data as illustrated in Figure 7. Using the data from this figure it is found that $\alpha = 88.5$. These data along with the elastic constants are sufficient to compute the response of the triaxial compression test with constant σ_3 . The parameters needed for analysis under general loading conditions can be computed from:

$$\phi_o^d = \frac{s_o}{\sigma_o + bs_o} \quad (206)$$

$$\alpha_1 = \frac{\sigma_o + bs_\infty}{\sigma_o + bs_o} \alpha \quad (207)$$

$$A_1 = \left(\frac{s_\infty}{\sigma_o + bs_\infty} - \phi_o^d \right) \alpha_1 \quad (208)$$

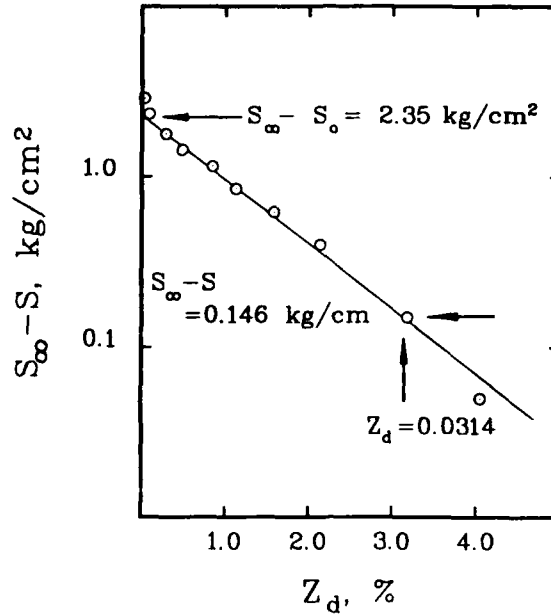


Figure 7: Semi-logarithmic plot of data for determination of constants.

Comparison to Experiment

72. A comparison between the theory and test data are shown in Figure 8. The shear strain e was computed from the definition of plastic strain given by Equation (209):

$$e^p = e - s/2\mu \quad (209)$$

where $2\mu = 1700 \text{ kg/cm}^2$ and e^p is found by integrating $de^p = F_d dz_d$ in view of Equations (202) and (199), and $F_d = \sigma$ to get:

$$e^p = (\sigma_0 + bs_\infty)z_d - \frac{b}{\alpha}(s_\infty - s_0)(1 - e^{-\alpha z_d}) \quad (210)$$

Thus the shear response can be expressed as two parametric equations in z_d . The volumetric response is obtained by numerically integrating Equations (189), (192), (193) and (195) to obtain e^p . The elastic volumetric strains during shear are computed using an elastic bulk modulus $K = 1133 \text{ kg/cm}^2$.

73. The comparison shown in Figure 8 is reasonably good for larger strains. At small strains, the error created by the one-term approximation is evident, especially in the initial shear-induced volume change where the plastic component of strain is underestimated. Note that by use of more exponential terms in Equation (200) the initial elastic response can be made as small as required to fit the data.

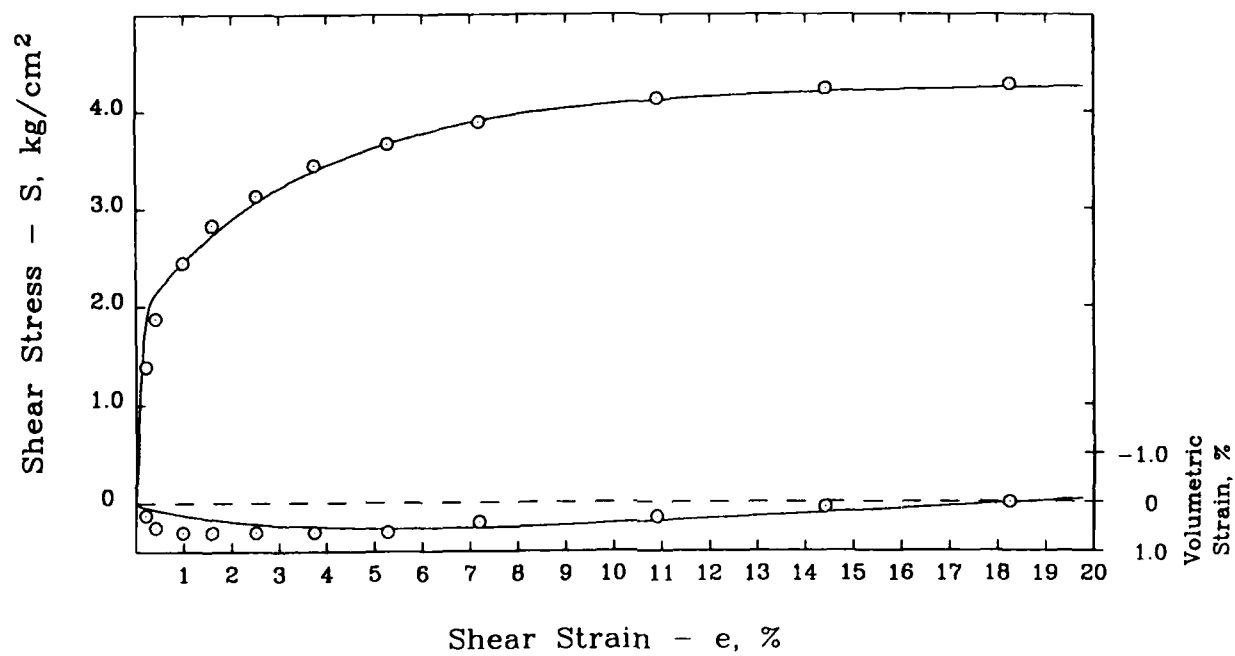


Figure 8: Comparison of one-exponential approximation with data.

PART V: CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

74. The work described in this report was intended to provide a theoretical framework on which to derive constitutive models for frictional materials based on linear thermodynamic relationships. The principal goal of this work was to develop a fundamentally sound model which correctly predicted both contraction and dilation caused by shearing. Two results drawn from this work are as follows:

- a. The contractive-dilative response of soil is captured through a straight-forward application of the theory of endochronic plasticity. In particular, dilatancy results from coupling terms in the rate equations. A significant finding is that coupling among internal mechanisms is through an averaging process that can be treated as an thermodynamic force which acts external to the hydrostatic mechanism. This is in contrast to the equally plausible hypothesis that there is an independent coupling between the shear and normal components of each mechanism.
- b. The model correctly predicts that contraction occurs immediately upon unloading even for the case where dilation occurs in the loading segment of a stress path. This is in direct agreement with experiments.

75. Future research should be directed as follows:

- a. The model should be extended to the case of undrained behavior. Procedurally, this could be done quite simply by applying the constraint $d\epsilon = nC_w du$ where C_w is the compressibility of pore fluid and du is the increment of pore pressure. The resulting relationships could then be used for analysis of undrained tests. Among the goals of this study would be
 - determination of properties from undrained tests
 - prediction of liquefaction
 - comparison of soil response for both drained and undrained conditions.
- b. A detailed study on property determination is needed. In particular, the determination of the shear-kernel function ϕ_d from data for complicated stress paths and generalized forms of F_d requires investigation.
- c. The dependence of elastic constants μ and K on stress and plastic strain requires investigation.
- d. The response of the model to non-proportional stress paths in deviatoric stress space and for loading under rotating principal stress axes requires investigation. The kinematic hardening behavior described by Equations (182) to (188) appears to be consistent with experimental data, reported by Alawaji et. al. (1986). That data, documented in detail by Sture, Alawi and Ko (1988), should be used to investigate the kinematic hardening mechanism further.

- e. The analysis needs to be extended to consider the effects of the third stress invariant.
- f. The model should be implemented in two and three dimensional numerical analysis codes for practical boundary value problems.
- g. The problem of uniqueness of solutions to boundary-value problems formulated from the model (already initiated by Valanis (1989)) should be studied further.

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APPENDIX A: DERIVATION OF STRESS-DILATANCY RELATIONSHIP

1. In this Appendix, the derivation of Equation (195) from (175) and (194) is described in detail. Relationships such as Equations (134), (140), (179), and (195) which express the *dilatancy rate* $d\epsilon^p/de^p$ to the shear-normal *stress ratio* s/σ are referred to as stress-dilatancy relationships in soil mechanics literature. The derivation of these relationships in the present analysis is based on combining the rate equations with the definition of endochronic time and solving the resulting simultaneous equations for $d\epsilon^p/de^p$. Whereas $d\epsilon^p/de^p$ appears as a quadratic term in the equations, two solutions are possible. Therefore, the main task of this appendix is to describe how the appropriate solution is chosen.

2. To reduce the complexity of the relevant equations, define the following:

$$dx = k d\epsilon^p \quad (A1)$$

$$dy = de^p \quad (A2)$$

$$a = \frac{F_h}{\sigma} \quad (A3)$$

$$b = \Gamma_o \frac{s}{\sigma} \quad (A4)$$

Write Equations (194) and (175) respectively as

$$a \frac{dx}{dz} + b \frac{dy}{dz} = 1 \quad (A5)$$

$$\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 = 1 \quad (A6)$$

where the solution is sought in terms of dx/dy . These equations may be interpreted geometrically as the intersection of a unit circle (Equation (A6)) by a line (Equation (A5)) as plotted on the cartesian axes dy/dz and dx/dz in Figure A1. The two solution points are S_1 and S_2 . The lines connecting the origin with these points, L_1 and L_2 , have slopes dy/dx which in view of Equations (A1) and (A2) are the two possible dilatancy rates.

3. The solutions to (A5) and (A6) can be determined upon substitution of Equation (175) to be

$$\frac{dx}{dy} = \frac{1 - b^2}{ab \pm \sqrt{a^2 + b^2 - 1}} \quad (A7)$$

With reference to Figure A1, S_1 corresponds to the positive solution whereas S_2 corresponds to the negative solution, that is

$$\frac{dx}{dy} = \frac{1 - b^2}{ab + \sqrt{a^2 + b^2 - 1}} \quad \text{for } S_1 \quad (\text{A8})$$

$$\frac{dx}{dy} = \frac{1 - b^2}{ab - \sqrt{a^2 + b^2 - 1}} \quad \text{for } S_2 \quad (\text{A9})$$

4. The line given by Equation (A1) crosses the dy/dz and dx/dz axes respectively at points P_1 and P_2 . The ordinate value of P_1 is $1/b$ and is a measure of the shear-normal stress ratio; at $s = 0$, $1/b$ is infinite. The demarcation between contraction and dilation (e.g. inequality (98)) lies on the circle at $b = 1$; thus *for the loading condition* dilatancy occurs when P_1 lies inside the circle.

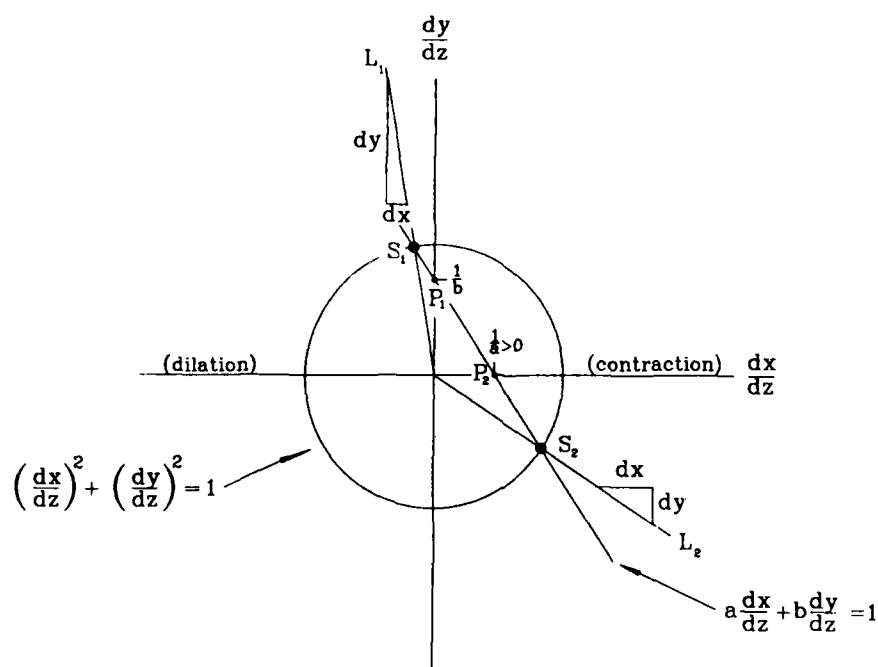
5. Point P_2 crosses the abscissa at $1/a$, which is a measure of the ratio between the prevailing hydrostatic stress and the hydrostatic yield stress. Under hydrostatic loading conditions, plastic yielding begins on the circle at $a = 1$. In general, $a \leq 1$. Therefore, P_2 lies on or within the circle.¹

6. When P_1 lies above the origin, the axial stress is greater than the lateral (confining) stress (commonly referred to as a triaxial compression test) and $\sigma_1 > \sigma_2 = \sigma_3 > 0$. When P_1 lies below the origin, the axial stress is less than the confining stress (triaxial extension) and $\sigma_1 = \sigma_2 > \sigma_3 > 0$. Points S_1 or S_2 falling above the abscissa correspond to solutions with increasing e^p whereas points falling below the abscissa correspond to decreasing values of e^p . A load reversal in a triaxial compression test will be characterized by solutions $dy < 0$ with $1/b > 0$; that is, a load reversal is depicted by P_1 above the origin and a solution point falling on the circle below the abscissa.

7. The solution for the initial condition ($s = 0$) in a triaxial test with $\sigma = \phi_e$ is shown in Figure A2. In this case, Equation (A5) describes a vertical line which is tangent to the circle at $(1/a, 0)$ giving as the only solution $de^p/dz = 1$ (or equivalently $de^p/de^p = \infty$). Two solutions are possible for the initial state when $a < 1$ as shown in Figure A3. The positive solution S_1 would be chosen for a triaxial compression test (e^p increasing) while the negative solution S_2 corresponds to the triaxial extension test (e^p decreasing).

8. Figures A4 and A5 illustrate the solution as s is increased in the triaxial compression test. For $1/b > 1$, both the positive and negative solutions correspond to contraction as both

¹In soil mechanics terminology, $a = 1$ corresponds to the normally consolidated state while $a < 1$ corresponds to an over consolidated state. A dense sand under low confining pressure or a highly over consolidated clay would have $a \ll 1$.



points of intersection (S_1 and S_2) fall to the right of the origin. The positive solution corresponds to loading while the negative solution applies upon unloading (when de^p changes sign). As s is increased such that $1/b < 1$, the positive (loading) solution falls to the left of the origin which corresponds to dilation. Here again, the negative (unloading) case corresponds to contraction. The extension test is similarly analyzed leading to the conclusion that dilation is the result of *loading* beyond $|1/b| < 1$, with contraction resulting otherwise. The various cases that may occur in a cyclic load history are illustrated by the sequence of sketches shown in Figure A6.

$$k \frac{d\epsilon^p}{|d\epsilon^p|} = \frac{1 - \left(\Gamma_o \frac{s}{\sigma}\right)^2}{\frac{F_h \Gamma_o |s|}{\sigma^2} \cos \psi + \sqrt{\left(\frac{F_h}{\sigma}\right)^2 + \left(\Gamma_o \frac{s}{\sigma}\right)^2} - 1} \quad (\text{A10})$$

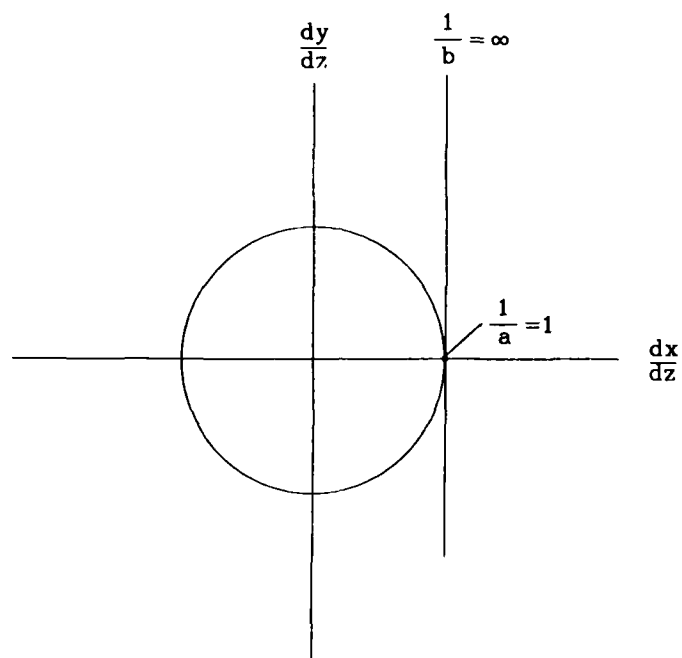


Figure A2: Solution for initial condition where $a = 1$ and $b = 0$ (normally consolidated case).

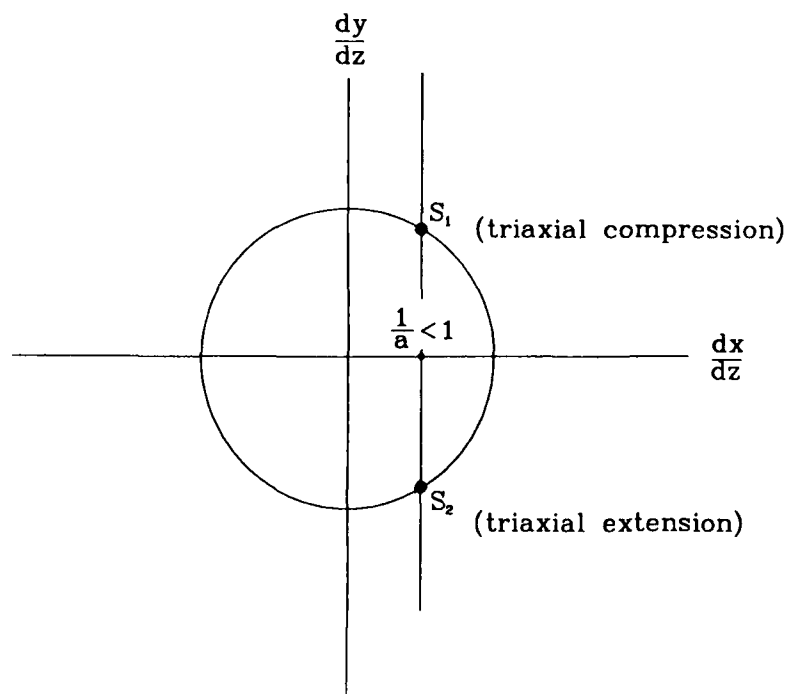


Figure A3: Solution for initial condition when $a < 0$ and $b = 0$ (over consolidated case).

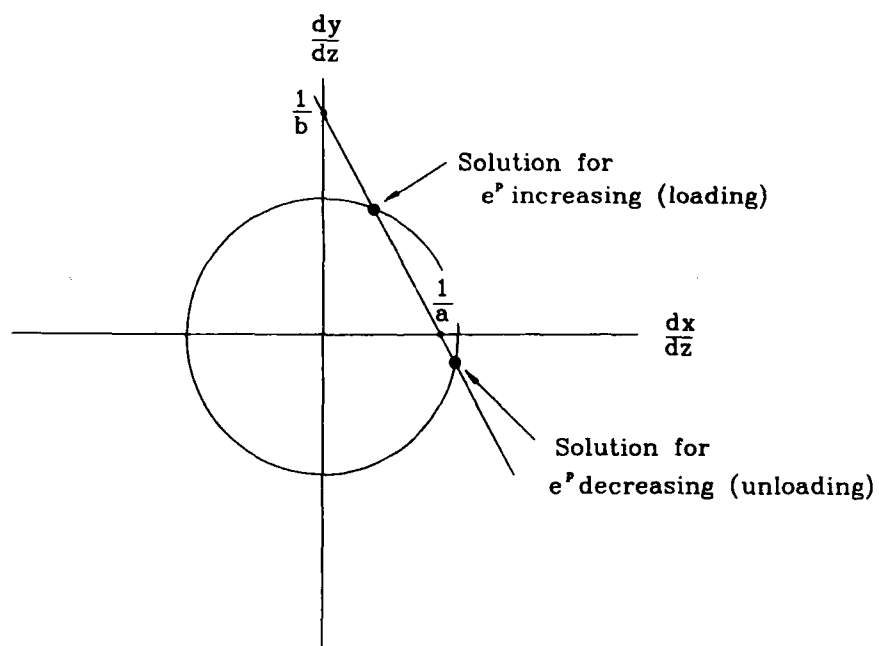


Figure A4: Solutions for contractive case for triaxial compression test.

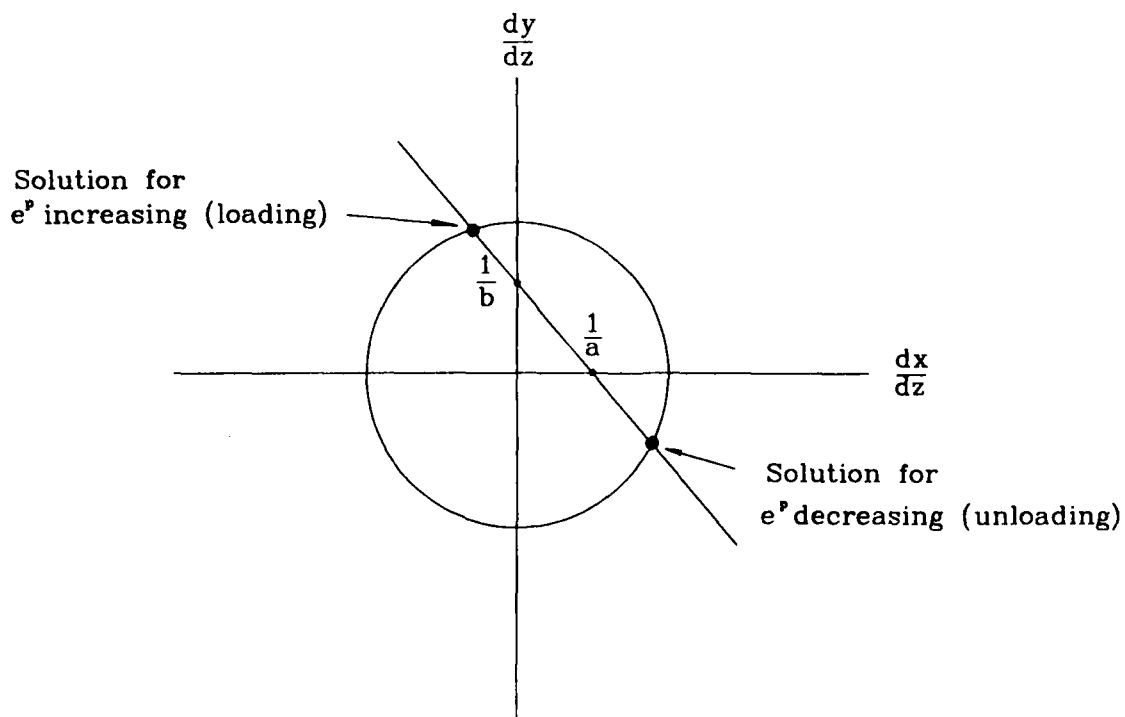


Figure A5: Solutions for dilative case in triaxial compression test.

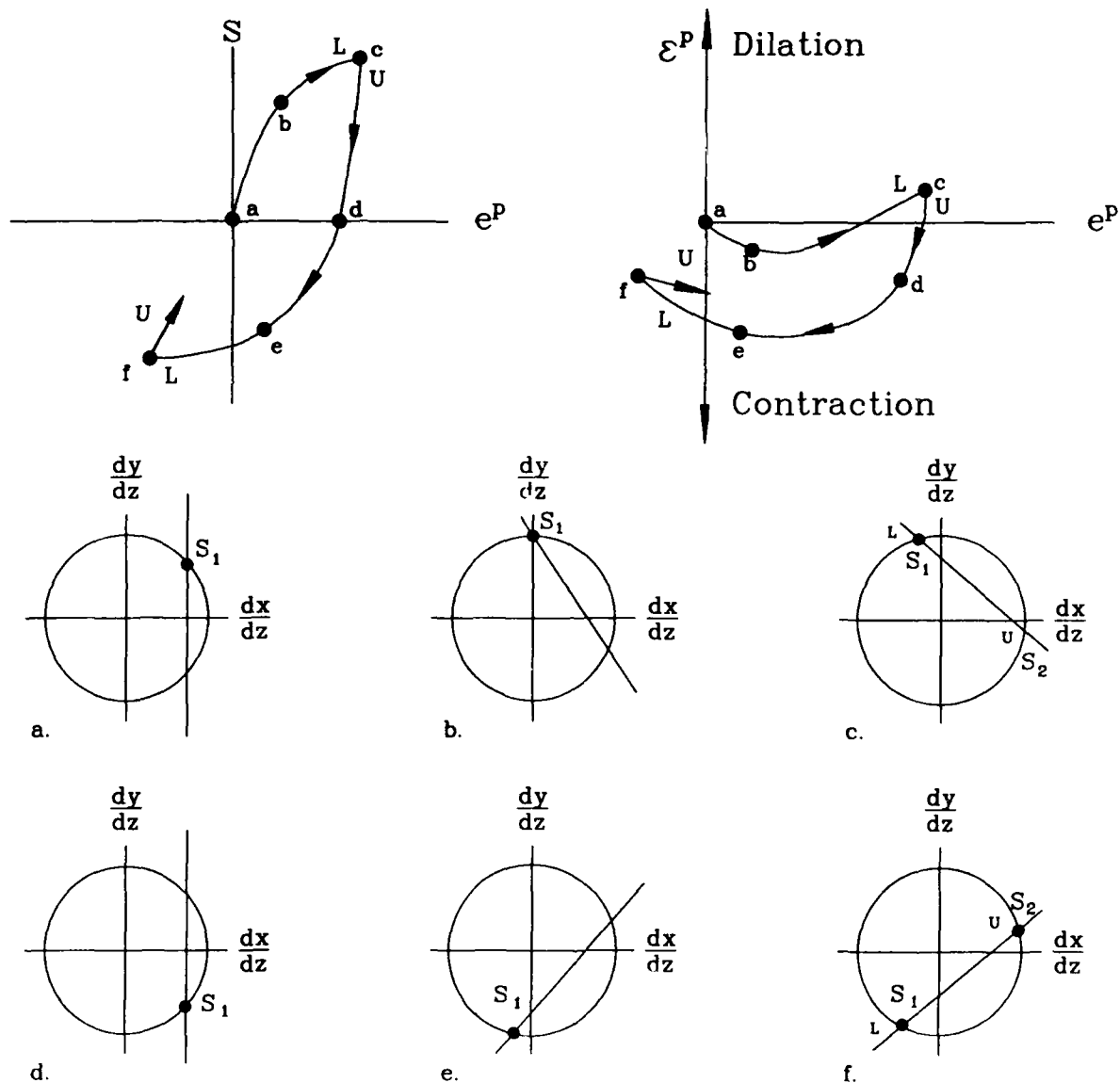


Figure A6: Correct solution (shown as solid point) for various points in compression-extension experiment of triaxial test.